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Translation and homothetical TH-surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ and Lorentzian-Minkowski space $\mathbb{E}^3_1$

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Abstract: In the 3-dimensional Euclidean space $\mathbb{E}^3$ and Lorentzian-Minkowski space $\mathbb{E}^3_1$, a translation and homothetical TH-surface is parameterized $z(u,v) = A(f(u)+g(v)) + Bf(u)g(v)$, where f and g are smooth functions and A, B are non-zero real numbers. In this paper, we define TH-surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ and Lorentzian-Minkowski space $\mathbb{E}^3_1$ and completely classify minimal or flat TH-surfaces.

Keywords: Translation surface, homothetical surface, minimal surface.

MSC: 51B20, 53A10, 53C45.

1. Introduction

The theory of minimal surfaces has found many applications in differential geometry and also in physics. In [1] and [2], H. Liu gave some classification results for translation surfaces. A minimal translation hypersurface in a Euclidean space is either locally a hyperplane or an open part of a cylinder on Scherk's surfaces, as proved in Dillen et al. [3]. In [1] was generalized to translation surfaces with constant mean curvature and constant Gaussian curvature in $\mathbb{E}^3$. Sağlam and Sabuncuoğlu proved that every homothetical lightlike hypersurface in a semi-Euclidean $\mathbb{E}^{m+2}_q$ space is minimal [4]. Jiu and Sun studied $n-$ dimensional minimal homothetical hypersurfces and gave their classification [5]. R. López [6] studied translation surfaces in the 3-dimensional hyperbolic space and classified minimal translation surfaces. Meng and Liu [7] considered factorable surfaces along two lightlike directions and spacelike-lightlike directions in Minkowski 3-space $\mathbb{E}^3_1$ and they also gave some classifcation theorems. In [8], Yu and Liu studied the factorable minimal surfaces in $\mathbb{E}^3_1$ and $\mathbb{E}^3$, and gave some classification theorems. Güler *et al.* [9] defined by translation and homothetical TH-surfaces in the three dimensional Euclidean space.

2. Preliminaries

Let $\mathbb{E}^3_1$ be a 3-dimensional Minkowski space with the scalar product of index 1 given by

$$g_L = ds^2 = -dx^2 + dy^2 + dz^2,$$

where (x, y, z) is a rectangular coordinate system of $\mathbb{E}^3_1$.

A vector V of $\mathbb{E}^3_1$ is said to be timelike if $g_L(V,V) < 0$, spacelike if $g_L(V,V) > 0$ or $V = 0$ and lightlike or null if $g_L(V,V) = 0$ and $V \neq 0$. A surface in $\mathbb{E}^3_1$ is spacelike, timelike or lightlike if the tangent plane at any point is spacelike, timelike or lightlike, respectively.

The Lorentz scalar product of the vectors V and W is defined by $g_L(V,W) = -v_1w_1 + v_2w_2 + v_3w_3$, where $V = (v_1, v_2, v_3)$, $W = (w_1, w_2, w_3) \in \mathbb{E}^3_1$.

For any $V, W \in \mathbb{E}^3_1$, the pseudo-vector product of V and W is defined as follows:

$$V \wedge_L W = \big(-v_2w_3 + v_3w_2,\ v_3w_1 - v_1w_3\ , v_1w_2 - v_2w_1 \big).$$

We denote a surface M^2 in $\mathbb{E}_1^3$ by

$$r(u,v) = \big(r_1(u,v),\ r_2(u,v),\ r_3(u,v)\big).$$

Definition 1 ([10])**.** A translation surface in Minkowski 3-space is a surface that is parameterized by either

$$\begin{aligned} r(u,v) &= (u,\ v,\ f(u)+g(v)) && \text{if } L \text{ is timelike,}\\ r(u,v) &= (f(u)+g(v),\ u,\ v) && \text{if } L \text{ is spacelike,}\\ r(u,v) &= (u+v,\ g(v),\ f(u)+v) && \text{if } L \text{ is lightlike,}\end{aligned}$$

with L the intersection of the two planes that contain the curves that generate the surface.

Theorem 2 ([11])**.** *i) The only translation surfaces with constant Gauss curvature $K = 0$ are cylindrical surfaces.*
ii) There are no translation surfaces with constant Gauss curvature $K \neq 0$ if one of the generating curves is planar.

Definition 3. A homothetical (factorable) surface M^2 in the 3-dimensional Lorentzian space $\mathbb{E}_1^3$ is a surface that is a graph of a function

$$z(u,\ v) = f(u)g(v),$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ and $g : J \subset \mathbb{R} \to \mathbb{R}$ are two smooth functions.

Theorem 4 ([11])**.** *Planes and helicoids are the only minimal homothetical surfaces in Euclidean space.*

Accordingly, we define an extended surface in $\mathbb{E}_1^3$ using definitions as above and called it TH-type surface as follows [9]:

Definition 5. A surface M^2 in the 3-dimensional Lorentzian space $\mathbb{E}_1^3$ is a TH-surface if it can be parameterized either by a patch

$$r(u,\ v) = (u,\ v,\ A(f(u)+g(v)) + Bf(u)g(v)) \tag{1}$$

or

$$r(u,\ v) = (A(f(u)+g(v)) + Bf(u)g(v),\ u,\ v), \tag{2}$$

where A and B are non-zero real numbers.

Remark 1. i) If $A \neq 0$ and $B = 0$ in (1), then surface is a translation surface.
ii) If $A = 0$ and $B \neq 0$ in (1), then surface is a homothetical (factorable) surface.

Let $\mathbf{N}$ denotes the unit normal vector field of M^2 and put $g_L(\mathbf{N},\mathbf{N}) = \varepsilon = \pm 1$, so that $\varepsilon = -1$ or $\varepsilon = 1$ according to M^2 is endowed with a Lorentzian or Riemannian metric, respectively.

The mean curvature and the Gauss curvature are

$$H = \frac{EN + GL - 2FM}{2\,|EG - F^2|},\ K = g_L(\mathbf{N},\mathbf{N})\frac{LN - M^2}{EG - F^2},$$

where E, G, F are the coefficients of the first fundamental form, L, M, N are the coefficients of the second fundamental form.

In this paper, we define TH-surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ and Lorentzian-Minkowski space $\mathbb{E}_1^3$, and completely classify minimal or flat TH-surfaces.

3. Minimal TH-surfaces in $\mathbb{E}_1^3$

A surface M^2 in the 3-dimensional Lorentzian space $\mathbb{E}_1^3$ is called minimal when locally each point on the surface has a neighborhood which is the surface of least area with respect to its boundary [12]. In 1775, J. B. Meusnier showed that the condition of minimality of a surface in $\mathbb{E}^3$ is equivalent with the vanishing of its mean curvature function, $H = 0$.

Let $z = f(x,y)$ define a graph M^2 in the Euclidean 3-space $\mathbb{E}^3$. If M^2 is minimal, the function f satisfies

$$(1 + {f_y}^2)f_{xx} - 2f_{xy}f_xf_y + (1 + {f_x}^2)f_{yy} = 0, \tag{3}$$

which was obtained by J. L. Lagrange in 1760.

Let M^2 be a TH-surface in $\mathbb{E}_1^3$ parameterized by a patch

$$r(u,\, v) = (u,\, v,\, A(f(u) + g(v)) + Bf(u)g(v)),$$

where A and B are non-zero real numbers.

So

$$r_u = (1,\, 0,\, f'\alpha),\;\; r_v = (0,\, 1,\, g'\gamma),$$

where $\alpha = A + Bg$ and $\gamma = A + Bf$.

After eliminating f' and g' we find

$$E = \frac{\gamma'^2\alpha^2 - B^2}{B^2},\;\; F = \frac{\alpha\gamma\alpha'\gamma'}{B^2},\; G = \frac{\gamma^2\alpha'^2 + B^2}{B^2}.$$

The unit normal vector is given by

$$\boldsymbol{N} = \frac{1}{WB}(\alpha\gamma',\; -\gamma\alpha',\, B),$$

where $W^2 = B^{-2}g_L(\boldsymbol{N},\boldsymbol{N})(\gamma'^2\alpha^2 - \alpha'^2\gamma^2 - B^2)$ and

$$g_L(\boldsymbol{N},\boldsymbol{N}) = \varepsilon,\;\; \varepsilon = \begin{cases} 1 & M^2 \text{ is spacelike } (\gamma'^2\alpha^2 - \alpha'^2\gamma^2 - B^2 > 0), \\ -1 & M^2 \text{ is timelike } (\gamma'^2\alpha^2 - \alpha'^2\gamma^2 - B^2 < 0). \end{cases}$$

The constant ε is called the sign of M^2.

The coefficients of the second fundamental form are given by

$$L = \frac{\alpha\gamma''}{BW},\;\; M = \frac{\alpha'\gamma'}{BW},\;\; N = \frac{\gamma\alpha''}{BW}.$$

The expression of H is

$$\begin{aligned} H &= \frac{B^2(\alpha f''(1 + g'^2\gamma^2) - 2B\alpha\gamma f'^2 g'^2 + \gamma g''(f'^2\alpha^2 - 1))}{2W^3} \\ &= \frac{\alpha\gamma''(B^2 + \alpha'^2\gamma^2) - 2\alpha\gamma\alpha'^2\gamma'^2 + \gamma\alpha''(\gamma'^2\alpha^2 - B^2)}{2BW^3}. \end{aligned} \tag{4}$$

Then M^2 is a minimal surface if and only if

$$\alpha\gamma''(B^2 + \alpha'^2\gamma^2) - 2\alpha\gamma\alpha'^2\gamma'^2 + \gamma\alpha''(\gamma'^2\alpha^2 - B^2) = 0. \tag{5}$$

We distinguish the following cases.

Case 1. Let $\gamma' = 0$. In this case (5) gives $\gamma\alpha'' = 0$.

i) If $\gamma = 0$, then $f(u) = -\frac{A}{B}$, M^2 is the horizontal plane of equation $z = -\frac{A^2}{B}$.
ii) If $\alpha'' = 0$, then $\alpha(v) = a_1 v + b_1$, $a_1,\, b_1 \in \mathbb{R}$, and $\gamma(u) = c_1$, $c_1 \in \mathbb{R}$, M^2 is the plane of equation $z = c_2 v + c_3$, $c_2,\, c_3 \in \mathbb{R}$.

Case 2. Let $\alpha' = 0$. In this case (5) gives $\gamma''\alpha = 0$.

i) If $\alpha = 0$, then $g(v) = -\frac{A}{B}$, M^2 is the horizontal plane of equation $z = -\frac{A^2}{B}$.
ii) If $\gamma'' = 0$, then $\gamma(u) = a_2 u + b_2$, $a_2,\, b_2 \in \mathbb{R}$, and $\alpha(v) = c_4$, $c_4 \in \mathbb{R}$, M^2 is the plane of equation $z = c_5 u + c_6$, $c_5,\, c_6 \in \mathbb{R}$.

Case 3. Let $\gamma'' = 0$ and $\gamma' \neq 0$. Then $\gamma(u) = \lambda u + \delta$, $(\lambda,\, \delta) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}$ and α is a solution of the following **ODE**

$$-2\lambda^2\alpha\alpha'^2 + \alpha''(\lambda^2\alpha^2 - B^2) = 0. \tag{6}$$

Then the general solution of (6) is given by

$$\alpha(v) = -\frac{B}{\lambda}\coth(\lambda_1 v + \lambda_2),\; \lambda_1,\, \lambda_2 \in \mathbb{R}.$$

Hence
$$g(v) = -\frac{1}{\lambda}\coth(\lambda_1 v + \lambda_2) - \frac{A}{B}, \ \lambda_1, \ \lambda_2 \in \mathbb{R}.$$

Case 4. Let $\alpha'' = 0$ and $\alpha' \neq 0$. Then $\alpha(v) = \lambda v + \delta$, $(\lambda, \delta) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}$ and γ is a solution of the following **ODE**
$$-2\lambda^2\gamma\gamma'^2 + \gamma''(\lambda^2\gamma^2 + B^2) = 0. \tag{7}$$

Then the general solution of (7) is given by
$$\gamma(u) = \frac{B}{\lambda}\tan(\lambda_1 u + \lambda_2), \ \lambda_1, \ \lambda_2 \in \mathbb{R}.$$

Hence
$$f(u) = \frac{1}{\lambda}\tan(\lambda_1 u + \lambda_2) - \frac{A}{B}, \ \lambda_1, \ \lambda_2 \in \mathbb{R}.$$

Case 5. Let $\gamma'' \neq 0$. By symmetry in the discussion of the case, we also suppose $\alpha'' \neq 0$. If we divide (5) by $\alpha\gamma\alpha'^2\gamma'^2$, we obtain
$$\frac{B^2\gamma''}{\gamma\alpha'^2\gamma'^2} + \frac{\gamma\gamma''}{\gamma'^2} - \frac{B^2\alpha''}{\alpha\alpha'^2\gamma'^2} + \frac{\alpha\alpha''}{\alpha'^2} - 2 = 0.$$

Thus, after a derivation with respect to u, followed by a derivation with respect to v, we obtain
$$\left(\frac{\gamma''}{\gamma\gamma'^2}\right)_{,u}\left(\frac{1}{\alpha'^2}\right)_{,v} - \left(\frac{\alpha''}{\alpha\alpha'^2}\right)_{,v}\left(\frac{1}{\gamma'^2}\right)_{,u} = 0.$$

Hence we deduce the existence of a real number $k \in \mathbb{R}$ such that
$$\begin{cases} \left(\frac{\gamma''}{\gamma\gamma'^2}\right)_{,u} = k\left(\frac{1}{\gamma'^2}\right)_{,u} \\ \left(\frac{\alpha''}{\alpha\alpha'^2}\right)_{,v} = k\left(\frac{1}{\alpha'^2}\right)_{,v}. \end{cases} \tag{8}$$

The first equation of (8) can integrate obtaining
$$\gamma'' = \gamma(k + c\gamma'^2). \tag{9}$$

From the second equation in (8), we obtain
$$\alpha'' = \alpha(k + b\alpha'^2). \tag{10}$$

Substituting the above in (5), we get
$$\alpha\gamma((k + c\gamma'^2)(B^2 + \alpha'^2\gamma^2) - 2\alpha'^2\gamma'^2 + (k + b\alpha'^2)(\gamma'^2\alpha^2 - B^2)) = 0.$$

If we simplify by $\alpha\gamma$ and then we divide by $\alpha'^2\gamma'^2$, we get
$$\frac{bB^2 - k\gamma^2}{\gamma'^2} - c\gamma^2 + 2 = \frac{cB^2 + k\alpha^2}{\alpha'^2} + b\alpha^2.$$

Hence, we deduce the existence of a real number $\lambda \in \mathbb{R}$ such that
$$\begin{cases} \gamma'^2 = \frac{bB^2 - k\gamma^2}{\lambda - 2 + c\gamma^2} \\ \alpha'^2 = \frac{cB^2 + k\alpha^2}{\lambda - b\alpha^2}. \end{cases} \tag{11}$$

Differentiating with respect to u and v, respectively, we have
$$\begin{cases} \gamma'' = -\frac{\gamma((\lambda-2)k + bcB^2)}{(\lambda - 2 + c\gamma^2)^2} \\ \alpha'' = \frac{\alpha(\lambda k + bcB^2)}{(\lambda - b\alpha^2)^2}. \end{cases} \tag{12}$$

Let us compare these expressions of α'' and γ'' with those ones that appeared in (9) and (10) and replace the values of γ'^2 and α'^2 obtained in (11).
We get

$$\begin{cases} (\lambda k + bcB^2)(\lambda - 1 - b\alpha^2) = 0 \\ ((\lambda - 2)k + bcB^2)(\lambda - 1 + c\gamma^2) = 0. \end{cases}$$

We discuss all possibilities.

i) If

$$\begin{cases} \lambda k + bcB^2 = 0 \\ (\lambda - 2)k + bcB^2 = 0, \end{cases}$$

then $k = 0$ and $bc = 0$. Then (12) gives $\gamma'' = 0$ and $\alpha'' = 0$, a contradiction.

ii) If

$$\begin{cases} \lambda k + bcB^2 = 0 \\ c = 0 \\ \lambda = 1, \end{cases}$$

we obtain $k = 0$. Then (12) gives $\gamma'' = 0$ and $\alpha'' = 0$, a contradiction.

iii) If

$$\begin{cases} (\lambda - 2)k + bcB^2 = 0 \\ b = 0 \\ \lambda = 1, \end{cases}$$

we obtain $k = 0$. Then (12) gives $\gamma'' = 0$ and $\alpha'' = 0$, a contradiction.

iv) If

$$\begin{cases} \lambda - 1 - b\alpha^2 = 0 \\ \lambda - 1 + c\gamma^2 = 0 \end{cases}$$

we deduce that α, γ are both constant functions, and so, $\gamma'' = 0$ and $\alpha'' = 0$, a contradiction.

v) If $b = 0$, $c = 0$ and $\lambda = 1$, Equation (11) writes as

$$\begin{cases} \gamma'^2 = k\gamma^2 \\ \alpha'^2 = k\alpha^2. \end{cases} \tag{13}$$

The equations (13) have the following solutions

$$\alpha(v) = k_1 e^{\sqrt{k}v}, \ \ \gamma(u) = k_2 e^{\sqrt{k}u}, \ k > 0,$$

where $k_1, k_2 \in \mathbb{R}$ are integration constants.

Hence

$$g(v) = \lambda_1 e^{\sqrt{k}v} - \frac{A}{B}, \ \ f(u) = \lambda_2 e^{\sqrt{k}u} - \frac{A}{B}, \ k > 0.$$

Therefore, we have the following:

Theorem 6. *Let M^2 be a TH-surface in $\mathbb{E}^3_1$. If M^2 is minimal surface, then M^2 can be parameterized as*

$$r(u, v) = (u, \ v, \ A(f(u) + g(v)) + Bf(u)g(v)),$$

where

1) *either $f(u) = -\frac{A}{B}$ and $g(v)$ is a smooth function in v.*
2) *$g(v) = -\frac{A}{B}$ and $f(u)$ is a smooth function in u.*
3) *$f(u) = \lambda_1 u + \lambda_2$ and $g(v) = \lambda_3 \coth(\lambda_4 v + \lambda_5) - \lambda_6$, $\lambda_i \in \mathbb{R}$.*
4) *$f(u) = \frac{1}{\lambda}\tan(\lambda_1 u + \lambda_2) - \frac{A}{B}$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $g(v) = \delta_5 v + \delta_6$, $\delta_i \in \mathbb{R}$.*
5) *$f(u) = \lambda_2 e^{\sqrt{k}u} - \frac{A}{B}$ and $g(v) = \lambda_1 e^{\sqrt{k}v} - \frac{A}{B}$.*

Let M^2 be a TH-surface in $\mathbb{E}^3_1$ parameterized by a patch

$$r(u, \ v) = (A(f(u) + g(v)) + Bf(u)g(v), \ u, \ v),$$

where A and B are non-zero real numbers.
So

$$r_u = (f'\alpha,\ 1,\ 0),\ \ r_v = (g'\gamma,\ 0,\ 1),$$

where $\alpha = A + Bg$ and $\gamma = A + Bf$.
We have

$$E = \frac{-\gamma'^2\alpha^2 + B^2}{B^2},\ \ F = -\frac{\alpha\gamma\alpha'\gamma'}{B^2},\ G = \frac{-\gamma^2\alpha'^2 + B^2}{B^2}.$$

The coefficients of the second fundamental form on M^2 are obtained by

$$L = \frac{\alpha\gamma''}{BW},\ \ M = \frac{\alpha'\gamma'}{BW},\ \ N = \frac{\gamma\alpha''}{BW}.$$

Then M^2 is a minimal surface if and only if

$$\alpha\gamma''(B^2 - \alpha'^2\gamma^2) + 2\alpha\gamma\alpha'^2\gamma'^2 - \gamma\alpha''(\gamma'^2\alpha^2 - B^2) = 0, \tag{14}$$

where $\alpha = A + Bg$ and $\gamma = A + Bf$.
Using the same algebraic techniques as in the case of surfaces (1), we get:

Theorem 7. *Let M^2 be a TH-surface in $\mathbb{E}^3_1$. If M^2 is minimal surface, then M^2 can be parameterized as*

$$r(u,\ v) = (A(f(u) + g(v)) + Bf(u)g(v),\ u,\ v),$$

where
1) *either* $f(u) = \frac{\zeta}{B}u + \alpha$ *and* $g(v) = -\frac{1}{\zeta}\coth(\lambda_3 v + \lambda_4) - \frac{A}{B}$.
2) $f(u) = -\frac{A}{B}$ *and* $g(v)$ *is a smooth function in* v.
3) $g(v) = -\frac{A}{B}$ *and* $f(u)$ *is a smooth function in* u.
4) *or* $g(v) = \frac{\delta}{B}v + \mu$ *and* $f(u) = -\frac{1}{\delta}\coth(\lambda_1 u + \lambda_2) - \frac{A}{B}$.

4. TH-surfaces with zero Gaussian curvature in $\mathbb{E}^3_1$

A non-degenerate surface in $\mathbb{E}^3_1$ is called flat, if its Gaussian curvature vanishes identically.

A surface in $\mathbb{E}^3_1$ parameterized by (1), after eliminating f, g and their derivatives, has Gaussian curvature

$$K = g_L(\mathbf{N},\mathbf{N})\frac{\alpha\gamma\alpha''\gamma'' - \gamma'^2\alpha'^2}{B^2W^4}.$$

Suppose that M^2 has zero Gaussian curvature. Then we have

$$\alpha\gamma\alpha''\gamma'' - \gamma'^2\alpha'^2 = 0. \tag{15}$$

Case 1. Let $\gamma' = 0$. In this case γ is a constant function $\gamma(u) = u_0$ and the parametrization of (1) writes as

$$r(u,v) = (u,\ v,\ \delta_1 g(v) + \delta_2);\ \delta_1, \delta_2 \in \mathbb{R}.$$

This means that M^2 is a cylindrical surface with base curve a plane curve in the $vz-$ plane.
Case 2. Let $\alpha' = 0$. In this case α is a constant function $\alpha(v) = v_0$ and the parametrization of (1) writes as

$$r(u,v) = (u,\ v,\ \delta_3 f(u) + \delta_4);\ \delta_3, \delta_4 \in \mathbb{R}.$$

This means that M^2 is a cylindrical surface with base curve a plane curve in the $uz-$ plane.
Case 3. Let $\gamma'' = 0$ and $\gamma' \neq 0$. Then $\gamma(u) = \lambda_1 u + \lambda_2$, $(\lambda_1, \lambda_2) \in \mathbb{R}\backslash \{0\} \times \mathbb{R}$. Moreover, (15) gives $\alpha' = 0$ and $\alpha(v) = v_0$ is a constant function. Now M^2 is the plane of equation $z(u,v) = \lambda_3 u + \lambda_4$; $\lambda_3, \lambda_4 \in \mathbb{R}$.
Case 4. Let $\alpha'' = 0$ and $\alpha' \neq 0$. Then $\alpha(v) = \lambda v + \delta_1$, $(\lambda, \delta_1) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}$. Moreover, (15) gives $\gamma' = 0$ and $\gamma(u) = u_0$ is a constant function. Now M^2 is the plane of equation $z(u,v) = \lambda_5 u + \lambda_6$; $\lambda_5, \lambda_6 \in \mathbb{R}$.

Case 5. Let $\gamma'' \neq 0$ and $\alpha'' \neq 0$.
Equation (15) writes as

$$\frac{\gamma\gamma''}{\gamma'^2} = \frac{\alpha'^2}{\alpha\alpha''}.$$

Therefore, there exists a real number $\lambda \in \mathbb{R} \setminus \{0\}$ uch that

$$\frac{\gamma\gamma''}{\gamma'^2} = \lambda = \frac{\alpha'^2}{\alpha\alpha''}.$$

Integrate these equations

$$\begin{cases} \gamma' = k_1\gamma^\lambda \\ \alpha' = k_2\alpha^{\frac{1}{\lambda}}, \end{cases} \tag{16}$$

where k_1 and k_2 are constants of integration.

i) If $\lambda = 1$, the general solution of (16) is given by

$$\begin{cases} \gamma(u) = \lambda_1 e^{k_1 u} \\ \alpha(v) = \lambda_2 e^{k_2 v}, \end{cases}$$

where λ_1 and λ_2 are constants of integration.
Hence

$$\begin{cases} f(u) = \lambda_3 e^{k_1 u} + \lambda_4 \\ g(v) = \lambda_5 e^{k_2 v} + \lambda_6, \end{cases}$$

where $\lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \mathbb{R}$.

ii) If $\lambda \neq 1$, the general solution of (16) is given by

$$\begin{cases} \gamma(u) = ((1-\lambda)k_1 u + c_1)^{\frac{1}{1-\lambda}} \\ \alpha(v) = ((\frac{\lambda-1}{\lambda})k_2 v + c_2)^{\frac{\lambda}{\lambda-1}}, \end{cases}$$

where c_1 and c_2 are constants of integration.
Hence

$$\begin{cases} f(u) = c_3((1-\lambda)k_1 u + c_1)^{\frac{1}{1-\lambda}} + c_4 \\ g(v) = c_5((\frac{\lambda-1}{\lambda})k_2 v + c_2)^{\frac{\lambda}{\lambda-1}} + c_6, \end{cases}$$

where $c_3, c_4, c_5, c_6 \in \mathbb{R}$.

Theorem 8. *Let M^2 be a TH-surface in $\mathbb{E}^3_1$ with constant Gauss curvature K. If M^2 has zero Gaussian curvature, then M^2 can be parameterized as*

$$r(u,v) = (u,\ v,\ z(u,v) = A(f(u) + g(v)) + Bf(u)g(v)),$$

where

1) *either* $f(u) = \lambda_1 e^{k_1 u} + \lambda_2$ *and* $g(v) = \lambda_3 e^{k_2 v} + \lambda_4$,
2) *or* $f(u) = \mu_1 u + \mu_2$ *and* $g(v) = \mu_3$,
3) *or* $g(v) = \nu_1 v + \nu_2$ *and* $f(u) = \nu_3$,
4) *or* $f(u) = \zeta_1((1-\lambda)k_1 u + \zeta_2)^{\frac{1}{1-\lambda}} + \zeta_3$ *and* $g(v) = \zeta_4((\frac{\lambda-1}{\lambda})k_2 v + \zeta_5)^{\frac{\lambda}{\lambda-1}} + \zeta_6$.

5. Minimal TH-surfaces in $\mathbb{E}^3$

Let M^2 be a TH-surface in the Euclidean 3-space $\mathbb{E}^3$. Then, M^2 is parameterized by

$$r(u,\ v) = (u,\ v,\ A(f(u) + g(v)) + Bf(u)g(v)),$$

where A and B are non-zero real numbers.
We have the natural frame $\{r_u,\ r_v\}$ given by

$$r_u = (1,\ 0,\ f'\alpha),\ \ r_v = (0,\ 1,\ g'\gamma),$$

where $\alpha = A + Bg$ and $\gamma = A + Bf$.
From this, the unit normal vector field $\boldsymbol{N}$ of M^2 is given by

$$\boldsymbol{N} = \frac{1}{W}(-\alpha f',\ -\gamma g',\ 1),$$

where $W = \sqrt{1 + f'^2\alpha^2 + g'^2\gamma^2}$.
The coefficients of the first fundamental form of M^2 are given by

$$E = 1 + f'^2\alpha^2,\ \ G = 1 + g'^2\gamma^2,\ \ F = f'g'\alpha\gamma.$$

The coefficients of the second fundamental form of the surface are

$$L = \frac{\alpha f''}{W},\ \ M = \frac{Bf'g'}{W},\ \ N = \frac{\gamma g''}{W}.$$

Hence, the mean curvature H and the Gaussian curvature K are given by, respectively

$$H = \frac{\alpha f''(1 + g'^2\gamma^2) - 2B\alpha\gamma f'^2 g'^2 + \gamma g''(1 + f'^2\alpha^2)}{2W^3}, \tag{17}$$

$$K = \frac{\alpha\gamma f'' g'' - B^2 f'^2 g'^2}{EG - F^2}. \tag{18}$$

If the surface is minimal, that is, $H = 0$ on M^2, we have from (17)

$$\alpha f''(1 + g'^2\gamma^2) - 2B\alpha\gamma f'^2 g'^2 + \gamma g''(1 + f'^2\alpha^2) = 0.$$

The previous equation may be rewritten as

$$\alpha\gamma''(B^2 + \alpha'^2\gamma^2) - 2\alpha\gamma\alpha'^2\gamma'^2 + \gamma\alpha''(B^2 + \gamma'^2\alpha^2) = 0. \tag{19}$$

Since the roles of α and γ in (19) are symmetric, we only discuss the cases according to the function γ. We distinguish cases.
Case 1. Let $\gamma' = 0$. In this case (19) gives $B^2\gamma\alpha'' = 0$.

i) If $\gamma = 0$, then $f(u) = -\frac{A}{B}$, M^2 is the horizontal plane of equation $z = -\frac{A^2}{B}$.
ii) If $\alpha'' = 0$, then $g(v) = av + b$, $a, b \in \mathbb{R}$, and $f(u) = c$, $c \in \mathbb{R}$, M^2 is the plane of equation $z = c_1 v + c_2$, c_1, $c_2 \in \mathbb{R}$.

Case 2. Let $\gamma'' = 0$ and $\gamma' \neq 0$, and by symmetry, $\alpha' \neq 0$. Then $\gamma(u) = \lambda u + \delta_1$, $(\lambda, \delta) \in \mathbb{R}^* \times \mathbb{R}$ and α is a solution of the following **ODE**

$$-2\lambda^2\alpha\alpha'^2 + \alpha''(B^2 + \lambda^2\alpha^2) = 0. \tag{20}$$

Then the general solution of (20) is given by

$$\alpha(v) = \frac{B}{\lambda}\tan(\lambda_1 v + \lambda_2),\ \lambda_1,\ \lambda_2 \in \mathbb{R}.$$

Hence

$$g(v) = \frac{1}{\lambda}\tan(\lambda_1 v + \lambda_2) - \frac{A}{B},\ \lambda_1,\ \lambda_2 \in \mathbb{R}.$$

So, the parametrization of M^2 can be written in the form

$$r(u,\ v) = (u,\ v,\ \lambda_3 u + \delta_2 + \frac{A}{\lambda}\tan(\lambda_1 v + \lambda_2) - \frac{A^2}{B} + B(\lambda_3 u + \delta_2)(\frac{1}{\lambda}\tan(\lambda_1 v + \lambda_2) - \frac{A}{B})),$$

where $(\lambda_3, \delta_2) \in \mathbb{R}^* \times \mathbb{R}$.
Case 3. Let $\gamma'' \neq 0$. By symmetry in the discussion of the case, we also suppose $\alpha'' \neq 0$. If we divide (19) by $\alpha\gamma\alpha'^2\gamma'^2$, we obtain

$$\frac{B^2\gamma''}{\gamma\alpha'^2\gamma'^2} + \frac{\gamma\gamma''}{\gamma'^2} + \frac{B^2\alpha''}{\alpha\alpha'^2\gamma'^2} + \frac{\alpha\alpha''}{\alpha'^2} - 2 = 0.$$

Thus, after a derivation with respect to u, followed by a derivation with respect to v, we obtain

$$\left(\frac{\gamma''}{\gamma\gamma'^2}\right)_{,u}\left(\frac{1}{\alpha'^2}\right)_{,v}+\left(\frac{\alpha''}{\alpha\alpha'^2}\right)_{,v}\left(\frac{1}{\gamma'^2}\right)_{,u}=0.$$

Hence we deduce the existence of a real number $k\in\mathbb{R}$ such that

$$\begin{cases}\left(\frac{\gamma''}{\gamma\gamma'^2}\right)_{,u}=k\left(\frac{1}{\gamma'^2}\right)_{,u}\\ \left(\frac{\alpha''}{\alpha\alpha'^2}\right)_{,v}=-k\left(\frac{1}{\alpha'^2}\right)_{,v}.\end{cases}\tag{21}$$

The first equation of (21) can integrate obtaining

$$\gamma''=\gamma(k+b_1\gamma'^2).\tag{22}$$

From the second equation in (21), we obtain

$$\alpha''=-\alpha(k+b_2\alpha'^2).\tag{23}$$

Substituting the above in (19), we get

$$\alpha\gamma((k+b_1\gamma'^2)(B^2+\alpha'^2\gamma^2)-2\alpha'^2\gamma'^2-(k+b_2\alpha'^2)(B^2+\gamma'^2\alpha^2))=0.$$

If we simplify by $\alpha\gamma$ and then we divide by $\alpha'^2\gamma'^2$, we get

$$\frac{k\gamma^2-b_2B^2}{\gamma'^2}-2+b_1\gamma^2=\frac{k\alpha^2-b_1B^2}{\alpha'^2}+b_2\alpha^2.$$

Hence, we deduce the existence of a real number $\lambda\in\mathbb{R}$ such that

$$\begin{cases}\gamma'^2=\frac{k\gamma^2-b_2B^2}{\lambda+2-b_1\gamma^2}\\ \alpha'^2=\frac{k\alpha^2-b_1B^2}{\lambda-b_2\alpha^2}.\end{cases}\tag{24}$$

Differentiating with respect to u and v, respectively, we have

$$\begin{cases}\gamma''=\frac{\gamma(\lambda k+2k-b_1b_2B^2)}{(\lambda+2-b_1\gamma^2)^2}\\ \alpha''=\frac{\alpha(\lambda k-b_1b_2B^2)}{(\lambda-b_2\alpha^2)^2}.\end{cases}\tag{25}$$

Let us compare these expressions of α'' and γ'' with those ones that appeared in (22) and (23) and replace the value of γ'^2 and α'^2 obtained in (24). We get

$$(\lambda k+2k-b_1b_2B^2)(1+\lambda-b_1\gamma^2)=0,$$

$$(\lambda k-b_1b_2B^2)(\lambda-1-b_2\alpha^2)=0.$$

We discuss all possibilities.

i) If $\lambda k+2k-b_1b_2B^2=0$ and $\lambda k-b_1b_2B^2=0$, then $k=0$ and $b_1b_2=0$. Then (25) gives $\gamma''=0$ and $\alpha''=0$, a contradiction.
ii) If $\lambda k+2k-b_1b_2B^2=0$, $\lambda=1$ and $b_2=0$, we obtain $k=0$. Then (25) gives $\gamma''=0$ and $\alpha''=0$, a contradiction.
iii) If $\lambda k-b_1b_2B^2=0$, $\lambda=-1$ and $b_1=0$, we obtain $k=0$. Then (25) gives $\gamma''=0$ and $\alpha''=0$, a contradiction.
iv) If $1+\lambda-b_1\gamma^2=0$ and $\lambda-1-b_2\alpha^2=0$, we deduce that α,γ are both constant functions, and so, $\gamma''=0$ and $\alpha''=0$, a contradiction.

Therefore, we have the following:

Theorem 9. *Let M^2 be a TH-surface in $\mathbb{E}^3$. If M^2 is minimal surface, then M^2 is plane or parameterized as*

$$r(u,v) = (u,\ v,\ A(f(u)+g(v)) + Bf(u)g(v)),$$

where

i) *either* $f(u) = \frac{\lambda_1}{B}u + \frac{\lambda_2 - A}{B}$ *and* $g(v) = \frac{1}{\lambda_1}\tan(\lambda_3 v + \lambda_4) - \frac{A}{B}$ *or*
ii) $f(u) = \frac{1}{\lambda_1}\tan(\lambda_2 u + \lambda_3) - \frac{A}{B}$ *and* $g(v) = \frac{\lambda_1}{B}v + \frac{\lambda_4 - A}{B}$.

6. TH-surfaces with zero Gaussian curvature in $\mathbb{E}^3$

A surface in Euclidean 3-space parameterized by (1) has Gaussian curvature

$$K = \frac{\alpha\gamma f''g'' - B^2 f'^2 g'^2}{EG - F^2}.$$

Hence that if $K = 0$, then

$$\alpha\gamma\alpha''\gamma'' - \gamma'^2\alpha'^2 = 0. \tag{26}$$

Since the roles of the function γ and α are symmetric in (26), we discuss the different cases according the function γ.

Case 1. Let $\gamma' = 0$. In this case γ is a constant function $\gamma(u) = u_0$ and the parametrization of (1) writes as

$$r(u,v) = (u,\ v,\ \delta_1 g(v) + \delta_2).$$

This means that M^2 is a cylindrical surface with base curve a plane curve in the $vz-$ plane.

Case 2. Let $\gamma'' = 0$ and $\gamma' \neq 0$. Then $\gamma(u) = \lambda u + \delta_1$, $(\lambda, \delta) \in \mathbb{R}^* \times \mathbb{R}$. Moreover, (26) gives $\alpha' = 0$ and $\alpha(v) = v_0$ is a constant function. Now M^2 is the plane of equation $z(u,v) = \lambda u + \delta_1$, $\lambda, \delta_1 \in \mathbb{R}$.

Case 3. Let $\gamma'' \neq 0$. By the symmetry on the arguments, we also suppose $\alpha'' \neq 0$.
Equation (26) writes as

$$\frac{\gamma\gamma''}{\gamma'^2} = \frac{\alpha'^2}{\alpha\alpha''}.$$

Therefore, there exists a real number $\lambda \in \mathbb{R}^*$ such that

$$\frac{\gamma\gamma''}{\gamma'^2} = \lambda = \frac{\alpha'^2}{\alpha\alpha''}.$$

Integrate these equations

$$\begin{cases} \gamma' = k_1\gamma^\lambda \\ \alpha' = k_2\alpha^{\frac{1}{\lambda}}, \end{cases} \tag{27}$$

where k_1 and k_2 are constants of integration.

i) If $\lambda = 1$, the general solution of (27) is given by

$$\begin{cases} \gamma(u) = \lambda_1 e^{k_1 u} \\ \alpha(v) = \lambda_2 e^{k_2 v}, \end{cases}$$

where λ_1 and λ_2 are constants of integration.
Hence

$$\begin{cases} f(u) = \lambda_3 e^{k_1 u} + \lambda_4 \\ g(v) = \lambda_5 e^{k_2 v} + \lambda_6, \end{cases}$$

where $\lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \mathbb{R}$.

i) If $\lambda \neq 1$, the general solution of (27) is given by

$$\begin{cases} \gamma(u) = ((1-\lambda)k_1 u + c_1)^{\frac{1}{1-\lambda}} \\ \alpha(v) = ((\frac{\lambda-1}{\lambda})k_2 v + c_2)^{\frac{\lambda}{\lambda-1}}, \end{cases}$$

where c_1 and c_2 are constants of integration.
Hence

$$\begin{cases} f(u) = c_3((1-\lambda)k_1u + c_1)^{\frac{1}{1-\lambda}} + c_4 \\ g(v) = c_5((\frac{\lambda-1}{\lambda})k_2v + c_2)^{\frac{\lambda}{\lambda-1}} + c_6, \end{cases}$$

where $c_3, c_4, c_5, c_6 \in \mathbb{R}$.

Theorem 10. *Let M^2 be a TH-surface in Euclidean $3-$ space $\mathbb{E}^3$ with constant Gauss curvature K. Then $K = 0$. Furthermore, the surface is plane or is a cylindrical surface over a plane curve or parameterized as*

$$r(u,v) = (u,\ v,\ A(f(u) + g(v)) + Bf(u)g(v)),$$

where

i) either $f(u) = \lambda_3 e^{k_1 u} + \lambda_4$ and $g(v) = \lambda_5 e^{k_2 v} + \lambda_6$ or
ii) $f(u) = c_3((1-\lambda)k_1u + c_1)^{\frac{1}{1-\lambda}} + c_4$ and $g(v) = c_5((\frac{\lambda-1}{\lambda})k_2v + c_2)^{\frac{\lambda}{\lambda-1}} + c_6$.

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