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Consistency Conditions of $f(R,G)$-Gravity Field Equations for Bianchi-Type III Metric

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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ABSTRACT

In this paper, we study the $f(R,G)$-gravitation theory under the assumption that the standard matter-energy content of the universe is a perfect fluid with linear barotropic equation of state within the framework of Bianchi-Type III model from the class of homogeneous and anisotropic universe models. However, whether such a restriction lead to any contradictions or inconsistencies in the field equations will create an issue that needs to be examined. Under the effective fluid approach, we will be concerned mainly the field equations in an orthonormal tetrad framework with an equimolar and examined the situation of establishing the functional form of $f(R,G)$ together with the scale factors, which are their solutions. Unlike similar studies, which are very few in the literature, instead of assuming preliminary solutions, we determined the consistency conditions of the field equations by assuming the matter energy content of the universe as an isotropic perfect fluid for Bianchi-Type III.

Keywords: Cosmology; $f(R,G)$-gravity; consistency conditions; Bianchi-Type III.

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1. INTRODUCTION

Immediately after 1905, when Einstein's Special Theory of Relativity was proposed, many attempts to put Newton's Gravitation Theory within the framework of this theory failed in the light of some criteria such as internal inconsistency, incompatibility with observations, and inaccuracy in predictions.

For this reason, the General Theory of Relativity (GRT), proposed by Einstein in 1916, describes gravitation in the language of the curvature of a 4-dimensional space-time manifold.

This theory both explained the centuries-old observational progression of the perihelian point of Mercury and predicted that the light passing around the Sun would be deflected due to the gravitation field of the Sun, and gave the correct amount of deflection. When compared with alternative theories such as the Brans-Dicke scalar-tensor theory and membrane-universe theories that emerged in the following years, it is seen that the GRT is still the most successful gravitation theory in events related to our solar system, that is, at the local scale. On the other hand; the application of the GRT field equations in the field of astrophysics and cosmology, that is, to much larger scales such as galaxies and intergalactic environments and the universe as a whole, has also contributed to obtaining results that are largely compatible with observations. For example: Big Bang, 2.7 K Cosmic microwave background (CMB) radiation, expansion of the universe, formation of large scale structures (galaxies), existence of black holes, etc. However, there are situations where the GRT or Relativist Cosmology based on it is insufficient. Based on observations, it is not yet sufficient to explain the problems such as the isotropisation of the universe, the flatness of the space, the horizon problem.

Let us now consider the astrophysical and cosmological observational developments that have made the validity of GRT at large scales questionable. The first of these is the issue called “Dark Matter” (DM). Observations on our Galaxy, initiated with Zwicki in the early 30’s, have raised some doubts that there is a missing mass in our Galaxy. Based on these observations, the ratio of DM, which should be in the universe, to the total matter is approximately %27.

The second issue is the surprising result obtained in 1998 from the observations of Supernova Type-Ia with high red-shift; because until then, it was believed that the universe has decelerating expansion, but these observations show that this was not the case; on the contrary, they have given results that the expansion occurs by accelerating. The amount of matter-energy measured for the universe cannot explain such an acceleration if it remains within the GRT's Relativist Cosmology. Because this "luminous matter", which we will call normal matter, is at most about %5 of the matter energy content of the universe, and this is almost one-fifteenth of the %73 required for acceleration.

Here, the remaining %68 of matter-energy is called "Dark Energy" (DE), and in order for this to cause the acceleration observed today, unlike the standard types of matter, it must be extremely negative pressure.

In fact, it is possible to explain the acceleration in question without assuming such an over pressurized exotic matter-energy when staying in GRT. The simplest candidate for this DE is the cosmological constant Λ, which corresponds to an effective fluid with a state parameter equal to -1, and therefore it was thought that Λ should be contained in the field equations. A second candidate for DE is various scalar fields imported into the Lagrangian of matter in the Einstein-Hilbert (EH) action. Such models are referred to as “quintessence” (=fifth element, scalar field). However, this creates a situation like explaining an unknown with another unknown that has never been observed until now. Moreover, the importation of such fields into the Einstein Field Equations (EFE) also creates improprieties in tests related to the solar system.

Both of the above approaches are aimed at changing the content of the energy-momentum tensor that forms the right-hand side of the EFE’s to create a matter-energy corresponding to DE. In Lagrangian formulation language, that means making appropriate changes to the matter Lagrangian.

The alternative to this approach is to modify the left side of the EFE, that is, to replace the geometric Lagrange, which is linear with respect to R in the EH-action, by an arbitrary function $f(R, G, R_{ab}R^{ab}, \dots)$ of the curvature invariants. The modified Einstein theory created in this way is called as $f(R, G, R_{ab}R^{ab}, \dots)$-gravity [1,2,3]. Here G is called as Gauss-Bonnet curvature invariant which is defined by a combination formed from the Riemann curvature tensor R_{abcd}

which are contractions of the $R_{ab} = R^{c}{}_{acb}$ Ricci curvature tensor and the $R = R^{c}{}_{c}$ Ricci curvature scalar as $G = R_{abcd}{}^{abcd} - 4R_{ab}R^{ab} + R^2$ [4,5,6,7,8]. In the literature; in order to simplify the equations of $f(R,G)$, Locally Rotationally Symmetric (LRS) Bianchi-Type models based on the assumption $B = C$ (or $A = B$ or $A = C$) are often considered to reduce the number of unknowns [9,10,11]. Apart from this simplification, which is often made in GRT for scale factors, one of the purposeful assumptions to obtain equations reduced to a single unknown is "to take the shear scalar proportional to the expansion scalar" [9,12,11]. Another assumption is to suggest some relations between the f function and a scale factor, such as the power-law [13,9,14,1,12]. Besides these, there are also purposeful assumptions about some cosmological parameters such as the deceleration parameter q and the Hubble parameter H [13,14,12]. FLRW models are very special models based on very high symmetry, such as being spatially homogeneous and isotropic. If the isotropic assumption is relaxed, then there are less symmetrical models that are spatially homogeneous but isotropic. These models are called Bianchi-type models. Mathematically, a Bianchi-type spacetime is a family of space-typed hypersurfaces that remain invariant under a 3 -parameter G_r $(r = 3)$ isometric group. The examination of the structure of the Lie algebra, which is connected to the G_3 isometric group, which acts on space-type hypersurfaces as a simple transition, was first discussed by Bianchi and classified as 9 different types called I,II,…,IX, which are not isomorphic to each other. These types have been used extensively since 1960, especially in order to reveal the effect of isomorphism[15].

In this study, the consistency conditions Eq.(2.19), Eq.(2.20) and Eq.(2.21) in the existence of the unified framework in which the total effective energy-momentum tensor is essentially supplied by an ideal fluid. It seems like a pretty strong assumption since not all sources in the field equation of a theory of gravity can be considered an ideal fluid. For example, in General Relativity we know that the source of the Kerr metrics not an ideal fluid [16].

The paper is organized as follows. We will present information about the mathematical material of calculating Einstein Field Equations and modified field equations for $f(R,G)$ in an orthonormal tetrad framework in the framework of Bianchi-Type III metric. After all, we will determine the conditions for consistency of the field equations with this constraint under the assumption of the matter energy content of the universe as an isometric perfect fluid, unlike the very few similar purposeful studies for Bianchi-Type III, rather than assuming preliminary solutions for the purpose taken as accepted in various articles in the literature.

2. METHODS OF CALCULATING $f(R,G)$-GRAVITY FIELD EQUATIONS

The diagonal form of the Bianchi-Type III space-time is considered in the form

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)e^{-2\lambda z}dy^2 + C^2(t)dz^2$$

where λ, is a real parameter with $\lambda \neq 0$ and $A(t)$, $B(t)$ and $C(t)$ are functions of cosmic time t.

Selection of EH action as

$$S = \frac{1}{2\kappa^2}\int_V d^4x\sqrt{-g}f(R,G) + S_m \qquad (2.1)$$

and theories with field equations obtained derived from here by taking variation according to the metric are called $f(R,G)$-gravity. Here, $-g > 0$ and $d^4x = \sqrt{-g}$ are the determinant of the metric with signature $+2$ and invariant volume element of 4-dimensional space-time, and respectively respectively S_m is the action of the Lagrange density L_m of the matter-energy in the form of $S_m = \int_V d^4\,x\sqrt{-g}L_m$. Taking variation of (1.1) with respect to the metric, that is, the field equations of $f(R,G)$-gravity is obtained in the following form:

$$f_R R_{ab} - \frac{1}{2}g_{ab}f + \frac{1}{2}g_{ab}Gf_G - \nabla_a\nabla_b f_R + g_{ab}\Box f_R - 4g_{ab}R^{cd}\nabla_c\nabla_d f_G - 4G_{ab}\Box f_G - 2R\nabla_a\nabla_b f_G + 4R_a{}^{c}\nabla_c\nabla_b f_G + 4R_b{}^{c}\nabla_c\nabla_a f_G - 4R_a{}^{cd}{}_b\nabla_c\nabla_d f_G = \kappa^2 T_{ab}{}^{m} \qquad (2.2)$$

In this variation, we define: $f_R(R,G) = \partial f(R,G)/\partial R$ and $f_G(R,G) = \partial f(R,G)/\partial G$, respectively [2].

Here, $T_{ab}{}^{m}$ is the energy-momentum tensor of the standard matter-energy fluid filling the universe. In order to be able to incorporate a variety of source terms in the Einstein Field Equations (EFE) [15], we use the standard decomposition of the stress-energy tensor T_{ab} with respect to a timelike vector field u ($u_a u^a = -1$), we can write standard matter-energy tensor T_{ab}, using the the superscript "m" as

$$T_{ab}{}^{m} = \mu^m u_a u_b + p^m h_{ab} + q_a{}^m u_b + q_b{}^m u_a + \pi_{ab}{}^m \tag{2.3}$$

where μ^m: energy density of standard matter, p^m: isotropic pressure, $q_a{}^m$: momentum density (energy flux), $\pi_{ab}{}^m$: anisotropic pressure, $q_a u^a = 0$, $\pi_{ab}{}^m u^b = 0$, $\pi_a{}^{ma} = 0$, $\pi_{ab}{}^m = \pi_{ba}{}^m$. Let's arrange the equation (2.2) to reveal the Einstein tensor. If we can write this equation in the form of

$$G_{ab} = \kappa^2 \left\{ \frac{T_{ab}{}^m}{f_R} + \frac{1}{\kappa^2 f_R} \left[\frac{1}{2} g_{ab}(f - Rf_R - Gf_G) + \nabla_a \nabla_b f_R - g_{ab} \Box f_R + 4 g_{ab} R^{cd} \nabla_c \nabla_d f_G + 4 G_{ab} \Box f_G + 2R \nabla_a \nabla_b f_G - 4 R_a{}^c \nabla_c \nabla_b f_G - 4 R_b{}^c \nabla_c \nabla_a f_G + 4 R_a{}^{cd}{}_b \nabla_c \nabla_d f_G \right] \right\} \tag{2.4}$$

and define the following two effective energy-momentum tensors

$$T_{ab}{}^{m.ef} \equiv \frac{T_{ab}{}^m}{f_R} \tag{2.5}$$

and

$$T_{ab}{}^{RG} \equiv \frac{1}{\kappa^2 f_R} \left[\frac{1}{2} g_{ab}(f - Rf_R - Gf_G) + \nabla_a \nabla_b f_R - g_{ab} \Box f_R + 4 g_{ab} R^{cd} \nabla_c \nabla_d f_G + 4 G_{ab} \Box f_G + 2R \nabla_a \nabla_b f_G - 4 R_a{}^c \nabla_c \nabla_b f_G - 4 R_b{}^c \nabla_c \nabla_a f_G + 4 R_a{}^{cd}{}_b \nabla_c \nabla_d f_G \right] \tag{2.6}$$

and also if we show their sum as

$$T_{ab}{}^{t.ef} \equiv T_{ab}{}^{m.ef} + T_{ab}{}^{RG} \tag{2.7}$$

then we obtain

$$G_{ab} = \kappa^2 T_{ab}{}^{t.ef} \tag{2.8}$$

$T_{ab}{}^{m.ef}$, which is defined by (2.5), is called effective matter-energy-momentum tensor. $T_{ab}{}^{RG}$, which is made up of all geometric terms, corresponds to the energy-momentum tensor of an effective fluid. $T_{ab}{}^{t.ef}$, which is the total effective energy-momentum tansor, is the sum of these two, reflecting the expression of some kind of interaction between standard matter and geometry. On the other hand, it can also be shown that these 4-type energy-momentum tensors satisfy the following conservation laws [15,17,18].

$$\nabla^a G_{ab} \equiv 0 \Rightarrow \nabla^a T_{ab}{}^{t.ef} = 0, \ \nabla^a T_{ab}{}^{m.ef} = 0, \ \nabla^a T_{ab}{}^{RG} = 0, \ \nabla^a T_{ab}{}^{m} = 0 \tag{2.9}$$

The above way of handling field equations is called the effective fluid approach

2.1 Applying 1+3 Covariant Decomposition

It is to do 1+3 decomposition of field equations in comoving orthonormal tetrad frame. This method; defines an comoving orthonormal tetrad frame of u^a and h_{ab} , a unit timelike vector field u determines a projection tensor h^{ab} according to $h^{ab} = g_{ab} + u^a u^b$, which at each point projects into the 3-space orthogonal to u. It follows that $h^c{}_a h_c{}^b = h_a{}^b$, $h_a{}^b u_b = 0$, $h^a{}_a = 3$.

For example,

$$u^a u^b G_{ab} = u^a u^b \kappa^2 T_{ab}{}^{t.ef} \quad \Rightarrow \quad u^0 u^0 G_{00} = \kappa^2 u^a u^b T_{ab}{}^{t.ef} \quad \Rightarrow \quad G_{00} = \kappa^2 \mu^{t.ef} \tag{2.10}$$

is obtained.

Here: $\mu^{t.ef}$ is the effective total matter-energy density and it is easily understood that it is given in terms of components of energy momentum tensors as

$$\mu^{t.ef} = T_{00}{}^{t.ef} = T_{00}{}^{m.ef} + T_{00}{}^{RG}$$
$$= \frac{\mu^m}{f_R} + \frac{1}{\kappa^2 f_R}\Big[-\frac{1}{2}(f - Rf_R - Gf_G) + \nabla_0\nabla_0 f_R + \Box f_R - 4R^{cd}\nabla_c\nabla_d f_G + 4G_{00}\Box f_G + 2R\nabla_0\nabla_0 f_G - 8R_0{}^c\nabla_c\nabla_0 f_G + 4R_0{}^{cd}{}_0\nabla_c\nabla_d f_G\Big]$$
$$= \frac{\mu^m}{f_R} + \mu^{RG} \qquad (2.11)$$

Similarly, the multiplication of both sides of the field equation by $\frac{1}{3}h^{ab}$ gives

$$\frac{1}{3}h^{ab}G_{ab} = \kappa^2\frac{1}{3}h^{ab}T_{ab}{}^{t.ef} \;\Rightarrow\; \frac{1}{3}(h^{11}G_{11} + h^{22}G_{22} + h^{33}G_{33}) = \kappa^2 p^{t.ef}$$
$$\Rightarrow\; \frac{1}{3}(G_{11} + G_{22} + G_{33}) = \kappa^2 p^{t.ef} \qquad (2.12)$$

where $p^{t.ef}$ is the effective total pressure and it is easily understood that it is given as

$$p^{t.ef} = \frac{1}{3}\left(T_{11}{}^{t.ef} + T_{22}{}^{t.ef} + T_{33}{}^{t.ef}\right) = \frac{1}{3}\left(T_{11}{}^{m.ef} + T_{22}{}^{m.ef} + T_{33}{}^{m.ef}\right) + \frac{1}{3}\left(T_{11}{}^{RG} + T_{22}{}^{RG} + T_{33}{}^{RG}\right)$$
$$= \frac{p^m}{f_R} + \frac{1}{\kappa^2 f_R}\Big[\frac{1}{2}(f - Rf_R - Gf_G) + \frac{1}{3}(\nabla_1\nabla_1 f_R + \nabla_2\nabla_2 f_R + \nabla_3\nabla_3 f_R) - \Box f_R + 4R^{cd}\nabla_c\nabla_d f_G + \frac{4}{3}(G_{11} + G_{22} + G_{33})\Box f_G + \frac{2}{3}R(\nabla_1\nabla_1 f_G + \nabla_2\nabla_2 f_G + \nabla_3\nabla_3 f_G) - \frac{8}{3}(R_1{}^c\nabla_c\nabla_1 f_G + R_2{}^c\nabla_c\nabla_2 f_G + R_3{}^c\nabla_c\nabla_3 f_G) + 4R_1{}^{cd}{}_1\nabla_c\nabla_d f_G) + \frac{4}{3}\left(4R_1{}^{cd}{}_1\nabla_c\nabla_d f_G + R_2{}^{cd}{}_2\nabla_c\nabla_d f_G + R_3{}^{cd}{}_3\nabla_c\nabla_d f_G\right)\Big]$$
$$= \frac{p^m}{f_R} + p^{RG} \qquad (2.13)$$

(Here, the property of $\pi_{11} + \pi_{22} + \pi_{33} = 0$ is used.)

Similarly, as a result of multiplying with the remaining two operators, similar relations can be obtained for the total effective heat flux $q_a^{t.ef}$ and the total effective anisotropic pressure tensor $\pi_a^{t.ef}$:

$$\mu^{t.ef}: \quad \frac{\dot A\dot B}{AB} + \frac{\dot B\dot C}{BC} + \frac{\dot C\dot A}{CA} - \frac{\lambda^2}{C^2} = \kappa^2\frac{\mu^m}{f_R} + \frac{1}{f_R}\Big\{-\frac{1}{2}(f - Rf_R - Gf_G) - \left(\frac{\dot A}{A} + \frac{\dot B}{B} + \frac{\dot C}{C}\right)\dot f_R + \left(-\frac{12\dot A\dot B\dot C}{ABC} + \frac{4\lambda^2}{C^2}\frac{\dot A}{A}\right)\dot f_G\Big\} \qquad (2.14)$$

$$p^{t.ef}: \quad -\frac{2}{3}\left(\frac{\ddot A}{A} + \frac{\ddot B}{B} + \frac{\ddot C}{C}\right) - \frac{1}{3}\left(\frac{\dot A\dot B}{AB} + \frac{\dot B\dot C}{BC} + \frac{\dot C\dot A}{CA}\right) + \frac{1}{3}\frac{\lambda^2}{C^2} = \kappa^2\frac{p^m}{f_R} + \frac{1}{f_R}\Big\{\frac{1}{2}(f - Rf_R - Gf_G) + \frac{2}{3}\left(\frac{\dot A}{A} + \frac{\dot B}{B} + \frac{\dot C}{C}\right)\dot f_R + \ddot f_R + \frac{4}{3}\left[\frac{\dot A}{A}\left(\frac{\ddot B}{B} + \frac{\ddot C}{C}\right) + \frac{\dot B}{B}\left(\frac{\ddot C}{C} + \frac{\ddot A}{A}\right) + \frac{\dot C}{C}\left(\frac{\ddot A}{A} + \frac{\ddot B}{B}\right)\right]\dot f_G + \frac{4}{3}\left[\left(\frac{\dot A\dot B}{AB} + \frac{\dot B\dot C}{BC} + \frac{\dot C\dot A}{CA}\right) - \frac{\lambda^2}{C^2}\right]\ddot f_G\Big\} \qquad (2.15)$$

$$q_1^{t.ef}: \quad 0 = 0 \qquad 2.16.a)$$

$q_2^{t.ef}$: $0 = 0$ $(2.16.b)$

$$q_3^{t.ef}: \quad \frac{\lambda}{C}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right) = -4\frac{\lambda}{C}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)\frac{\dot{A}}{A}\frac{\dot{f}_G}{f_R} \quad \Rightarrow \quad \frac{\lambda}{C}\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right)\left(1+\frac{4\dot{A}}{A}\frac{\dot{f}_G}{f_R}\right) = 0 \qquad (2.16.c)$$

$$\pi_{11}^{t.ef}: \quad \frac{1}{3}\left(\frac{2\ddot{A}}{A}-\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}\right)+\frac{1}{3}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{2\dot{B}\dot{C}}{BC}+\frac{\dot{C}\dot{A}}{CA}\right)+\frac{2}{3}\frac{\lambda^2}{C^2}$$
$$= \frac{1}{f_R}\left\{\frac{1}{3}\left(-\frac{2\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)\dot{f}_R - \frac{4}{3}\left[\frac{\dot{A}}{A}\left(\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}\right)+\frac{\dot{B}}{B}\left(-\frac{2\ddot{C}}{C}+\frac{\ddot{A}}{A}\right)+\frac{\dot{C}}{C}\left(\frac{\ddot{A}}{A}-\frac{2\ddot{B}}{B}\right)\right]\dot{f}_G - \frac{4}{3}\left(\frac{\dot{A}\dot{B}}{AB}-\frac{2\dot{B}\dot{C}}{BC}+\frac{\dot{C}\dot{A}}{CA}+\frac{2\lambda^2}{C^2}\right)\ddot{f}_G\right\} \qquad (2.17.a)$$

$$\pi_{22}^{t.ef}: \quad \frac{1}{3}\left(-\frac{\ddot{A}}{A}+\frac{2\ddot{B}}{B}-\frac{\ddot{C}}{C}\right)+\frac{1}{3}\left(\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}-\frac{2\dot{C}\dot{A}}{CA}\right)-\frac{1}{3}\frac{\lambda^2}{C^2}$$
$$= \frac{1}{f_R}\left\{\frac{1}{3}\left(\frac{\dot{A}}{A}-\frac{2\dot{B}}{B}+\frac{\dot{C}}{C}\right)\dot{f}_R - \frac{4}{3}\left[\frac{\dot{A}}{A}\left(\frac{\ddot{B}}{B}-\frac{2\ddot{C}}{C}\right)+\frac{\dot{B}}{B}\left(\frac{\ddot{C}}{C}+\frac{\ddot{A}}{A}\right)+\frac{\dot{C}}{C}\left(-\frac{2\ddot{A}}{A}+\frac{\ddot{B}}{B}\right)\right]\dot{f}_G - \frac{4}{3}\left(\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}-\frac{2\dot{C}\dot{A}}{CA}-\frac{\lambda^2}{C^2}\right)\ddot{f}_G\right\} \qquad (2.17.b)$$

$$\pi_{33}^{t.ef}: \quad \frac{1}{3}\left(-\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}+\frac{2\ddot{C}}{C}\right)+\frac{1}{3}\left(-\frac{2\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{C}\dot{A}}{CA}\right)-\frac{1}{3}\frac{\lambda^2}{C^2}$$
$$= \frac{1}{f_R}\left\{\frac{1}{3}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}-\frac{2\dot{C}}{C}\right)\dot{f}_R - \frac{4}{3}\left[\frac{\dot{A}}{A}\left(-\frac{2\ddot{B}}{B}+\frac{\ddot{C}}{C}\right)+\frac{\dot{B}}{B}\left(\frac{\ddot{C}}{C}-\frac{2\ddot{A}}{A}\right)+\frac{\dot{C}}{C}\left(\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}\right)\right]\dot{f}_G - \frac{4}{3}\left(-\frac{2\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{C}\dot{A}}{CA}-\frac{\lambda^2}{C^2}\right)\ddot{f}_G\right\} \qquad (2.17.c)$$

$\pi_{12}^{t.ef}$: $0 = 0$ $(2.17.d)$

$\pi_{23}^{t.ef}$: $0 = 0$ $(2.17.e)$

$\pi_{32}^{t.ef}$: $0 = 0$ $(2.17.f)$

Now, starting from the three equations in (2.17), let’s create the following organized three auxiliary equation with side by side subtractions and label arranged forms with "Π" symbols:

$$\Pi_{11}-\Pi_{22}:\quad \frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}+\frac{\dot{C}}{C}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)+\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\frac{\dot{f}_R}{f_R}+4\left[\frac{\dot{C}}{C}\left(\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}\right)+\frac{\ddot{C}}{C}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\right]\frac{\dot{f}_G}{f_R}+4\frac{\dot{C}}{C}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\frac{\ddot{f}_G}{f_R}+\frac{\lambda^2}{C^2}\left(1+\frac{4\ddot{f}_G}{f_R}\right)=0 \qquad (2.18.\mathrm{a})$$

$$\Pi_{22}-\Pi_{33}:\quad \frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}+\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)+\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)\frac{\dot{f}_R}{f_R}+4\left[\frac{\dot{A}}{A}\left(\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}\right)+\frac{\ddot{A}}{A}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)\right]\frac{\dot{f}_G}{f_R}+4\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)\frac{\ddot{f}_G}{f_R}=0 \qquad (2.18.\mathrm{b})$$

$$\Pi_{33}-\Pi_{11}:\quad \frac{\ddot{C}}{C}-\frac{\ddot{A}}{A}+\frac{\dot{B}}{B}\left(\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right)+\left(\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right)\frac{\dot{f}_R}{f_R}+4\left[\frac{\dot{B}}{B}\left(\frac{\ddot{C}}{C}-\frac{\ddot{A}}{A}\right)+\frac{\ddot{B}}{B}\left(\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right)\right]\frac{\dot{f}_G}{f_R}+4\frac{\dot{B}}{B}\left(\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right)\frac{\ddot{f}_G}{f_R}-\frac{\lambda^2}{C^2}\left(1+\frac{4\ddot{f}_G}{f_R}\right)=0 \qquad (1.18.\mathrm{c})$$

2.2 Consistency Conditions

Describing the ordinary matter-energy content of the universe with a perfect fluid with only $\mu^m(t)$ matter-energy density and $p^m(t)$ pressure, or in other words, the assumption of $q_\alpha^m \equiv 0$ and $\pi_{\alpha\beta}^m \equiv 0$ $(\alpha,\beta,\ldots=1{,}2{,}3)$ as the equation of state in the energy-momentum tensor of the fluid raises the question of whether the $G_{0\alpha}$ and $G_{\alpha\beta}$ components of the field equations will be consistent with this assumption. In the following, this problem will be examined in terms of consistency with the perfect fluid assumption.

Now, firstly let's consider the equation (2.16.c). For $\lambda \neq 0$, the cases where the consistency of this equation will be ensured are:

$$(i)\quad 1+4\frac{\dot{A}}{A}\frac{\dot{f}_G}{f_R}\neq 0 \quad and \quad \frac{\dot{C}}{C}-\frac{\dot{B}}{B}=0 \quad\Leftrightarrow\quad C=k_1B\ ,(k_1=constant>0) \qquad (2.19)$$

$$(j)\quad C\neq k_1B \quad and \quad 1+4\frac{\dot{A}}{A}\frac{\dot{f}_G}{f_R}=0 \qquad (2.20)$$

$$(k)\quad 1+4\frac{\dot{A}}{A}\frac{\dot{f}_G}{f_R}=0 \quad and \quad \frac{\dot{C}}{C}-\frac{\dot{B}}{B}=0 \quad\Leftrightarrow\quad C=k_1B \qquad (2.21)$$

Before discussing the reflection of these conditions, which we will call "primary conditions", to the equations (2.18), let's arrange these equations as follows. Writing Eq. (2.18.a) by taking common factors as

$$\left(\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}\right)\left(1+\frac{4\dot{C}}{C}\frac{\dot{f}_G}{f_R}\right)+\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\left(\frac{\dot{C}}{C}+\frac{\dot{f}_R}{f_R}+\frac{4\ddot{C}}{C}\frac{\dot{f}_G}{f_R}+\frac{4\dot{C}}{C}\frac{\ddot{f}_G}{f_R}\right)+\frac{\lambda^2}{C^2}\left(1+\frac{4\ddot{f}_G}{f_R}\right)=0$$

and also paying attention to that

$$\frac{\dot{C}}{C}+\frac{\dot{f}_R}{f_R}+\frac{4\ddot{C}}{C}\frac{\dot{f}_G}{f_R}+\frac{4\dot{C}}{C}\frac{\ddot{f}_G}{f_R}=\frac{1}{Cf_R}\left(\dot{C}f_R+C\dot{f}_R+4\ddot{C}\dot{f}_G+4\dot{C}\ddot{f}_G\right)=\frac{1}{Cf_R}\frac{d}{dt}\left(Cf_R+4\dot{C}\dot{f}_G\right)$$

we obtain, for (2.18. a)

$$\left(\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}\right)\frac{1}{Cf_R}\left(Cf_R+4\dot{C}\dot{f}_G\right)+\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\frac{1}{Cf_R}\frac{d}{dt}\left(Cf_R+4\dot{C}\dot{f}_G\right)+\frac{\lambda^2}{C^2}\left(1+\frac{4\ddot{f}_G}{f_R}\right)=0$$

or, multiplying by Cf_R, we get

$$\left(\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}\right)\left(Cf_R+4\dot{C}\dot{f}_G\right)+\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\frac{d}{dt}\left(Cf_R+4\dot{C}\dot{f}_G\right)+\frac{\lambda^2}{C}\left(f_R+4\ddot{f}_G\right)=0 \qquad (2.22.\mathrm{a})$$

Similarly, the equations (2.18.b) and (2.18.c) can be written as, respectively,

$$\left(\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}\right)\left(Af_R+4\dot{A}\dot{f}_G\right)+\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)\frac{d}{dt}\left(Af_R+4\dot{A}\dot{f}_G\right)=0 \qquad (2.22.\mathrm{b})$$

$$\left(\frac{\ddot{C}}{C}-\frac{\ddot{A}}{A}\right)\left(Bf_R+4\dot{B}\dot{f}_G\right)+\left(\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right)\frac{d}{dt}\left(Bf_R+4\dot{B}\dot{f}_G\right)-\frac{\lambda^2 B}{C^2}\left(f_R+4\ddot{f}_G\right)=0 \qquad (2.22.\mathrm{c})$$

In this context, it would be appropriate to draw attention to these features: These three equations above are the revised $\Pi_{11}-\Pi_{22}$, $\Pi_{22}-\Pi_{33}$ and $\Pi_{33}-\Pi_{11}$ equations, respectively. If one of them is identically zero, the remaining two equations become opposite signs of each other. Indeed, for example when $\Pi_{22}-\Pi_{33}\equiv 0$, then here; $\Pi_{22}=\Pi_{33}$ and therefore $\Pi_{11}-\Pi_{22}=\Pi_{11}-\Pi_{33}=-(\Pi_{33}-\Pi_{11})$. From this it follows that it would be sufficient for consistency to show that at least two of the equations (2.18) or (2.22) are satisifed identically. On the other hand, the equation satisfied identically, for example $\Pi_{11}-\Pi_{22}\equiv 0$ equation, will in no way result in $\Pi_{11}\equiv 0$ or $\Pi_{22}\equiv 0$ if additional information is not available. If other $\Pi_{22}-\Pi_{33}\equiv 0$ and $\Pi_{33}-\Pi_{11}\equiv 0$ are provided, it gives $\Pi_{11}=\Pi_{22}=\Pi_{33}=0$. However, if the property of $Tr(\Pi_{\alpha\beta})\equiv\Pi_{11}+\Pi_{22}+\Pi_{33}\equiv 0$ is taken into consideration, it is necessary to conclude that $\Pi_{11}=\Pi_{22}=\Pi_{33}\equiv 0$. In that case; The consistency of equations (2.18) is also equivalent to the consistency of equations (2.17. a,b,c).

Now, consider the primary condition (i). Under this condition (2.22.b) is satisfied identically; It can be easily seen that (2.22.c) is equal to (2.22.a) with opposite sign. So the only independent equation that should be examined under the primary condition (i) is (2.22.a). This equation containing four unknowns A, B, C and $f-$function, using the primary condition (i), reduced to the following equation as the equation with three unknowns:

$$\left(\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}\right)\left(Bf_R+4\dot{B}\dot{f}_G\right)+\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\frac{d}{dt}\left(Bf_R+4\dot{B}\dot{f}_G\right)+\frac{\lambda^2}{k_1 B}\left(f_R+4\ddot{f}_G\right)=0 \qquad (2.23)$$

At first glance, the situations in which this equation will be satisfied are as follows:

$$(i_1)\quad B=k_2 A \quad (k_2=constant>0) \quad \text{and} \quad f_R+4\ddot{f}_G=0 \qquad (2.24)$$

$$(i_2)\quad Bf_R+4\dot{B}\dot{f}_G=0 \quad \text{and} \quad f_R+4\ddot{f}_G=0 \qquad (2.25)$$

Henceforth, these will be called the "secondary condition". It is worth to point out here that the condition (i_1) and (i_2) are fulfilled together, that is, a condition such as "$B=k_2A$ and $Bf_R+4\dot{B}\dot{f}_G=0$ and $f_R+4\ddot{f}_G=0$" cannot be written because it would contradict the primary condition (i). Now, the condition (i_1) considering together with (i), is reduces to condition

$$C=k_1 B=k_3 A \quad (k_3=k_1k_2>0) \qquad (2.26)$$

Therefore the Bianchi-type III model reduces to a model with a single scale factor, such as the RW model. The condition (i_2) consists of two relations that are formed in a way that does not require a condition such as $B\propto A$ to be put forward. In this system of equations, by eliminating f_R, the following equation can be obtained

$$-B\ddot{f}_G+\dot{B}\dot{f}_G=0 \qquad \Rightarrow \qquad -\frac{\ddot{f}_G}{\dot{f}_G}+\frac{\dot{B}}{B}=0$$

Considering the relation $C=k_1B$ in the fundamental condition (i), K_1 and $K_2\equiv\frac{K_1}{k_1}$ being constants that can be positive or negative, integration of this last equation gives

$$\dot{f}_G = K_1 B = K_2 C \quad (2.27.a)$$

From this equation, both from the system of equations and by derivation the following relations are obtained

$$\ddot{f}_G = K_1 \dot{B} = K_2 \dot{C} \quad (2.27.b)$$

$$f_R = -4K_1 \dot{B} = -4K_2 \dot{C} \quad (2.27.c)$$

$$\dot{f}_R = -4K_1 \ddot{B} = -4K_2 \ddot{C} \quad (2.27.d)$$

$$\ddot{f}_R = -4K_1 \dddot{B} = -4K_2 \dddot{C} \quad (2.27.e)$$

Now, although its effect is not seen in advance, it may be thought to put forward another condition different from (i_1) and (i_2) for the fulfillment of equation (2.23) as

$$(i_3) \quad B \neq k_2 A \quad and \quad B f_R + 4\dot{B}\dot{f}_G \neq 0 \quad and \quad f_R + 4\ddot{f}_G = 0 \quad (2.28)$$

In this case (1.23) equation, provided that $B \neq k_2 A$ and $B f_R + 4\dot{B}\dot{f}_G \neq 0$, can be written in the form of

$$\frac{\ddot{A}B - A\ddot{B}}{\dot{A}B - A\dot{B}} + \frac{\frac{d}{dt}\left(B f_R + 4\dot{B}\dot{f}_G\right)}{B f_R + 4\dot{B}\dot{f}_G} = 0 \quad (2.29)$$

The integration of this equation yields

$$\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)\left(f_R + 4\frac{\dot{B}}{B}\dot{f}_G\right) = \frac{M}{AB^2} \quad (2.30)$$

where M is a constant of integration which can be positive or negative. A similar of this equation can be obtained by inserting $B \to C$, if desired, using the relation $C = k_1 B$ in the fundamental condition (i).

Now, let us consider the primary condition (j), provided that $C \neq k_1 B$. It can be expressed by the following equations which are equivalent to each other

$C \neq k_1 B$;

$$1 + 4\frac{\dot{A}}{A}\frac{\dot{f}_G}{f_R} = 0 \quad \Leftrightarrow \quad A f_R + 4\dot{A}\dot{f}_G = 0 \quad \Leftrightarrow \quad f_R = -4\frac{\dot{A}}{A}\dot{f}_G \quad \Leftrightarrow \quad \dot{f}_G = -\frac{1}{4}\frac{A}{\dot{A}} f_R \quad (2.31)$$

Under this condition, it is immediately seen that (2.22.b) is satisfied. In order to satisfy (2.22.a), at first glance, it seems that one of the following conditions will be sufficient.

$$(j_1) \quad B = k_2 A \quad (k_2 = constant > 0) \quad \text{and} \quad f_R + 4\ddot{f}_G = 0 \quad (2.32)$$

$$(j_2) \quad C f_R + 4\dot{C}\dot{f}_G = 0 \quad \text{and} \quad f_R + 4\ddot{f}_G = 0 \quad (2.33)$$

These will also called as “secondary condition”. Here again, let us note that a third secondary condition such as $B = k_2 A$ and $C f_R + 4\dot{C}\dot{f}_G = 0$ and $f_R + 4\ddot{f}_G = 0$, which corresponds to the situation in which the conditions (j_1) and (j_2) are met together cannot be put forward; because combining it with (j) leads to a result such as $B \propto A$, $C \propto A \Rightarrow C \propto B$, which contradicts the primary condition (j). Now, combining the secondary condition (j_1) with the primary condition (j) can be expressed by the system of equations

$$B f_R + 4\dot{B}\dot{f}_G = 0 \quad (2.34.a)$$

$$f_R + 4\ddot{f}_G = 0 \quad (2.34.b)$$

By first eliminating f_R between these two equations, it can be found

$$-B\ddot{f}_G + \dot{B}\dot{f}_G = 0$$

Integrating this equation and again using the secondary condition (j_1), give $\dot{f}_G$ and f_R and their derivatives with respect to time as follows

$$\dot{f}_G = K_1 B = K_2 A \quad (2.35.a)$$

$$\ddot{f}_G = K_1 \dot{B} = K_2 \dot{A} \quad (2.35.b)$$

$$f_R = -4K_1 \dot{B} = -4K_2 \dot{A} \quad (2.35.c)$$

$$\dot{f}_R = -4K_1\ddot{B} = -4K_2\ddot{A} \quad (2.35.\mathrm{d})$$

$$\ddot{f}_R = -4K_1\dddot{B} = -4K_2\dddot{A} \quad (2.35.\mathrm{e})$$

Here, K_1 is an arbitrary integration constant and K_2 is a new constant defined as $K_2 = K_1 k_2$. Now, it is immediately seen that the remaining equation (2.22.c) is identically satisfied under the secondary condition (j_1), so that it does not have to be $C \propto A$.

Now, let's consider the secondary condition (j_2). Comparing the first relation in (j_2) with (2.31), it is understood that $C = k_1 A$ and this relation causes equation (2.22.c) to be satisfied identically, without requiring any relation between B and A. The equations (2.32) correspond to the condition (j_2). To summarize the result of the examinations so far regarding the condition (j); with $C \neq k_1 B$, equations (2.35) are sufficient to satisfy equations (2.17.c) and (2.18) identically.

On the other hand, although it is not clear whether (2.22) equations will be satisfied together, let's impose the condition

$$(j_3) \qquad f_R + 4\ddot{f}_G = 0 \quad (2.36)$$

which is less restrictive than (j_1) and (j_2) under the primary condition (j).

In this case, from (j) and (j_3) it is obtained less restrictive relations than (2.35) and (2.36) as

$$\dot{f}_G = K_2 A \quad (2.37.\mathrm{a})$$

$$\ddot{f}_G = K_2 \dot{A} \quad (2.37.\mathrm{b})$$

$$f_R = -4K_2\dot{A} \quad (2.37.\mathrm{c})$$

$$\dot{f}_R = -4K_2\ddot{A} \quad (2.37.\mathrm{d})$$

$$\ddot{f}_R = -4K_2\dddot{A} \quad (2.37.\mathrm{e})$$

Putting these relations in equations (2.22.a), (2.22.b) and (2.22.c); (2.22.a) reduces to

$$\left(\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}\right)\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)+\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\left(\frac{\ddot{A}}{A}-\frac{\ddot{C}}{C}\right) = 0 \quad (2.38)$$

(2.22.b) is identically zero; and (2.22.c) gives the opposite sign of the above equation. If we can write Eq. (2.38) in the form of

$$\frac{\ddot{A}B - A\ddot{B}}{\dot{A}B - A\dot{B}} + \frac{\ddot{A}C - A\ddot{C}}{\dot{A}C - A\dot{C}} = 0$$

its integration gives

$$\left(\dot{A}B - A\dot{B}\right)\left(\dot{A}C - A\dot{C}\right) = L \quad (2.39.\mathrm{a})$$

Or

$$\left(\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\right)\left(\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right) = \frac{L}{A^2BC} \quad (2.39.\mathrm{b})$$

Here, L is an integration constant that can be positive or negative. As will be noted, this set of conditions does not create a condition for scale factors as $A \propto B \propto C$.

Now, finally, let's consider the primary condition (k). Under this condition, it can be easily seen that (2.22.b) is satisfied identically and (2.22.a) can be written as

$$\left(\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}\right)\left(Bf_R + 4\dot{B}\dot{f}_G\right) + \left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\frac{d}{dt}\left(Bf_R + 4\dot{B}\dot{f}_G\right) + \frac{\lambda^2}{k_1 B}\left(f_R + 4\ddot{f}_G\right) = 0 \quad (2.40)$$

This is satisfied in the following cases:

$$(k_1) \qquad B \propto A \quad \mathit{and} \quad f_R + 4\ddot{f}_G = 0 \quad (2.41)$$

$$(k_2) \qquad Bf_R + 4\dot{B}\dot{f}_G = 0 \quad \mathit{and} \quad f_R + 4\ddot{f}_G = 0 \quad (2.42)$$

$$(k_3) \qquad B \propto A \quad \mathit{and} \quad Bf_R + 4\dot{B}\dot{f}_G = 0 \quad \mathit{and} \quad f_R + 4\ddot{f}_G = 0 \quad (2.43)$$

When each of these secondary conditions is evaluated together with the primary condition (k), it can easily be found that they all lead to a single set of relations as follows:

$$C \propto B \propto A \quad (2.44.a)$$

Table 1. Consistency Conditions [18]

	$(i)\ \frac{\dot{C}}{C} - \frac{\dot{B}}{B} = 0\ (\Leftrightarrow C = k_1 B)\quad as\quad 1 + 4\frac{\dot{A}}{A}\frac{\dot{f}_G}{f_R} \neq 0$		
$(i_1)\ B = k_2 A\ and\ f_R + 4\ddot{f}_G = 0$ $C = k_1 B = k_3 A$	$(i_2)\ Bf_R + 4\dot{B}\dot{f}_G = 0\ and\ f_R + 4\ddot{f}_G = 0$ $(B \propto A\ \ not\ necessary)$ $\dot{f}_G = K_1 B = K_2 C$ $\ddot{f}_G = K_1 \dot{B} = K_2 \dot{C}$ $f_R = -4K_1\dot{B} = -4K_2\dot{C}$ $\dot{f}_R = -4K_1\ddot{B} = -4K_2\ddot{C}$	$(i_3)\ f_R + 4\ddot{f}_G = 0$ $(B \propto A\ \ not\ necessary)$ $B \neq kA\ and\ Bf_R + 4\dot{B}\dot{f}_G \neq 0$ $\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)\left(f_R + 4\frac{\dot{B}}{B}\dot{f}_G\right) = \frac{M}{AB^2}$ or $\left(\frac{\dot{A}}{A} - \frac{\dot{C}}{C}\right)\left(f_R + 4\frac{\dot{C}}{C}\dot{f}_G\right) = \frac{Mk_1^2}{AC^2}$	$\ddot{f}_R = -4K_1\dddot{B} = -4K_2\dddot{C}$
	$(j)\ \frac{\dot{C}}{C} - \frac{\dot{B}}{B} \neq 0\ (\Leftrightarrow C \neq k_1 B)\quad as\quad 1 + 4\frac{\dot{A}}{A}\frac{\dot{f}_G}{f_R} = 0$		
$(j_1)\ B = k_2 A\ and\ f_R + 4\ddot{f}_G = 0$ $(C \propto A\ \ not\ necessary)$ $\dot{f}_G = K_1 B = K_2 A$ $\ddot{f}_G = K_1\dot{B} = K_2\dot{A}$ $f_R = -4K_1\dot{B} = -4K_2\dot{A}$ $\dot{f}_R = -4K_1\ddot{B} = -4K_2\ddot{A}$ $\ddot{f}_R = -4K_1\dddot{B} = -4K_2\dddot{A}$	$(j_2)\ Cf_R + 4\dot{C}\dot{f}_G = 0\ and\ f_R + 4\ddot{f}_G = 0$ $C = K_1 A\ (B \propto A\ \ not\ necessary)$ $\dot{f}_G = K_1 C = K_2 A$ $\ddot{f}_G = K_1\dot{C} = K_2\dot{A}$ $f_R = -4K_1\dot{C} = -4K_2\dot{A}$ $\dot{f}_R = -4K_1\ddot{C} = -4K_2\ddot{A}$ $\ddot{f}_R = -4K_1\dddot{C} = -4K_2\dddot{A}$	$(j_3)\ f_R + 4\ddot{f}_G = 0$ $(A \propto B \propto C\ \ not\ necessary)$ $\dot{f}_G = K_2 A$ $\ddot{f}_G = K_2\dot{A}$ $f_R = -4K_2\dot{A}$ $\dot{f}_R = -4K_2\ddot{A}$ $\ddot{f}_R = -4K_2\dddot{A}$ $\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right)\left(\frac{\dot{C}}{C} - \frac{\dot{A}}{A}\right) = \frac{L}{A^2BC}$	
	$(k)\ \frac{\dot{C}}{C} - \frac{\dot{B}}{B} = 0(\Leftrightarrow C = k_1 B)\quad and\quad 1 + 4\frac{\dot{A}}{A}\frac{\dot{f}_G}{f_R} = 0$		
$(k_1)\ B \propto A\ and\ f_R + 4\ddot{f}_G = 0$	$(k_2)\ Bf_R + 4\dot{B}\dot{f}_G = 0$ $and\ f_R + 4\ddot{f}_G = 0$	$(k_3)\ B \propto A\ and\ Bf_R + 4\dot{B}\dot{f}_G = 0$ $and\ f_R + 4\ddot{f}_G = 0$ $for\ all$ $(A \propto B \propto C)$	
	$\dot{f}_G = K_1 A = K_2 B = K_3 C$ $\ddot{f}_G = K_1\dot{A} = K_2\dot{B} = K_3\dot{C}$		
		$f_R = -4K_1\dot{A} = -4K_2\dot{B} = -4K_3\dot{C}$ $\dot{f}_R = -4K_1\ddot{A} = -4K_2\ddot{B} = -4K_3\ddot{C}$ $\ddot{f}_R = -4K_1\dddot{A} = -4K_2\dddot{B} = -4K_3\dddot{C}$	

$$\dot{f}_G = K_1 A = K_2 B = K_3 C \quad (2.44.\mathrm{b})$$

$$\ddot{f}_G = K_1 \dot{A} = K_2 \dot{B} = K_3 \dot{C} \quad (2.44.\mathrm{c})$$

$$f_R = -4K_1 \dot{A} = -4K_2 \dot{B} = -4K_3 \dot{C} \quad (2.44.\mathrm{d})$$

$$\dot{f}_R = -4K_1 \ddot{A} = -4K_2 \ddot{B} = -4K_3 \ddot{C} \quad (2.44.\mathrm{e})$$

$$\ddot{f}_R = -4K_1 \dddot{A} = -4K_2 \dddot{B} = -4K_3 \dddot{C} \quad (2.44.\mathrm{f})$$

3. CONCLUSION

Studies in the literature include:

- The energy-momentum tensor is taken as a perfect fluid, and the work is carried out by resetting the anisotropic pressure and heat flux. However, the consistency conditions given by the reset are not addressed.
- In the studied metrics, local rotational symmetries and A(t)=B(t) ... are assumed and solutions are proposed.
- Solutions have been proposed by taking special cases in the f(R,G)-gravity function.

In this study, we obtained all the consistency conditions of the Bianchi-III model in detail, without taking the above specific assumptions and putting forward any restrictions. We discussed the $f(R,G)$ -gravity theory under the assumption that the normal (standard) matter-energy content of the universe is a perfect fluid with the linear barotropic state equation, in the framework of the Bianchi Type-III model, which is the class of homogeneous and anisotropic universe models in space. We have handled this study differently from the very few similar-purpose studies in the literature. Instead of assuming pre-solutions for the purpose in advance, as is done in similar studies, assuming the matter-energy content of the universe as an isometric perfect fluid, we determined the conditions for the consistency of the field equations in case. In our next studies, we will consider the reflection of the consistency conditions we found for the Bianchi Type-III model with the remaining equations. We show all the above conditions in Table 1.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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