Archives of Current Research International

15(2): 1-14, 2018; Article no.ACRI.44885
ISSN: 2454-7077

Traveling Waves with Critical Speed in a Delayed Diffusive Epidemic Model

Haimei Xu[1], Jiangbo Zhou[1*] and Liyuan Song[1]

[1]*Faculty of Science, Jiangsu University, Zhenjiang 212013, Jiangsu, P.R. China.*

Authors' contributions

This work was carried out in collaboration between all authors. Author JZ designed the study and corrected the final manuscript. Authors HX and LS managed the analyses of the study and wrote the first draft of the manuscript. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ACRI/2018/44885
Editor(s):
(1) Dr. Basak Taseli, Professor, Department of Environmental Engineering, Faculty of Engineering, Giresun University, Turkey.
Reviewers:
(1) Kejun Zhuang, Anhui University of Finance and Economics, P.R. China.
(2) S. K. Asha, Gulbarga University, India.
Complete Peer review History: http://www.sciencedomain.org/review-history/27304

Original Research Article

Received 10 September 2018
Accepted 14 November 2018
Published 19 November 2018

ABSTRACT

In a recent paper [K. Zhou, M. Han, Q. Wang, Math. Method. Appl. Sci. 40 (2016) 2772-2783], the authors investigated the traveling wave solutions of a delayed diffusive SIR epidemic model. When the basic reproduction number $R_0 > 1$ and the wave speed $c > c^*$ (c^* is the critical speed), they obtained the existence of a non-trivial and non-negative traveling wave solution. When $R_0 > 1$ and $0 < c < c^*$, they established non-existence of the non-trivial and non-negative traveling wave solutions. When $R_0 > 1$ and $c = c^*$, the existence of traveling waves was left as an open problem. The aim of this paper is to solve this problem by applying upper-lower solution method and Schauder's fixed point theorem.

Keywords: Diffusive epidemic model; traveling wave; reaction-diffusion equation; critical speed.

**Corresponding author: Email: ujszjb@126.com;*

1. INTRODUCTION

As is known to all, the spatial movement of the population plays a vital role in epidemic models and the time variation governs the dynamical behaviour of the epidemic propagation. Spreading processes of many epidemics have been modelled by reaction-diffusion equations [1,2]. Usually, the infected individuals cannot be transmitted the epidemic to others immediately after being infected, and a certain amount of biological development is necessary before they can infect others. Thus, time delay has been incorporated into many epidemic models [3-8]. Travelling waves in these models describe the epidemic wave moving out from an initial disease-free equilibrium to the endemic equilibrium with a constant speed [9]. The wave speed c measures how fast the disease invades geographically. Results on this topic may help people take necessary measures in advance to prevent the disease, or at least, decrease possible negative consequences.

Until recently, much attention has been paid to the existence of super-critical travelling wave solutions for diffusive epidemic models with or without time delay [2-19]. For example, Zhan et al. [19] investigated a spatially extended *SI* epidemic system with a nonlinear incidence rate. Using the method for the dynamical system, they obtained the existence of a heteroclinic orbit connecting two equilibrium points in R^3 which corresponds to a super-critical travelling wave solution connecting the disease-free and endemic equilibria for this *SI* epidemic system. Lotfi et al. [7] derived the existence of super-critical travelling waves of a delayed diffusive epidemic model with specific nonlinear incidence rate by constructing a pair of upper and lower solutions and applying the Schauder's fixed point theorem. Very recently, Zhou et al. [3] investigated the travelling wave solution of a delayed diffusive epidemic model

$$\begin{cases} S_t(x,t)=d_1S_{xx}(x,t)+\mu-\mu S(x,t)-\dfrac{\beta S(x,t)I(x,t-\tau)}{S(x,t)+I(x,t-\tau)}, \\ I_t(x,t)=d_2I_{xx}(x,t)+\dfrac{\beta S(x,t)I(x,t-\tau)}{S(x,t)+I(x,t-\tau)}-(\mu+\gamma)I(x,t), \end{cases} \tag{1.1}$$

where $S(x,t)$ and $I(x,t)$ are the densities of susceptible and infective individuals at location x and time t, respectively. The constants $d_i > 0$ $(i=1,2)$ denote the spatial motility of each class, $\mu > 0$ represents the natural death rate of each class, $\beta > 0$ is the transmission coefficient, $\gamma > 0$ refers to the recovery rate and $\sigma \geq 0$ is the latency of the infection. The corresponding reaction system of (1.1) is

$$\begin{cases} \dot{S}(t)=\mu-\mu S(t)-\dfrac{\beta S(t)I(t-\tau)}{S(t)+I(t-\tau)}, \\ \dot{I}(t)=\dfrac{\beta S(t)I(t-\tau)}{S(t)+I(t-\tau)}+(\mu+\gamma)I(t). \end{cases} \tag{1.2}$$

System (1.2) has a disease-free equilibrium $E^0=(1,0)$ and a unique endemic equilibrium $E^*=(S^*,I^*)$ if the basic reproduction number, where $R_0 := \dfrac{\beta}{\mu+\gamma} > 1,\ S^*=\dfrac{\mu}{\beta-\gamma}$, $I^*=\dfrac{\mu(\beta-\mu-\gamma)}{(\mu+\gamma)(\beta-\gamma)}$.

The traveling wave profile system of (1.1) is

$$\begin{cases} cS'(\eta)=d_1S''(\eta)+\mu-\mu S(\eta)-\dfrac{\beta S(\eta)I(\eta-c\tau)}{S(\eta)+I(\eta-c\tau)}, \\ cI'(\eta)=d_2I''(\eta)+\dfrac{\beta S(\eta)I(\eta-c\tau)}{S(\eta)+I(\eta-c\tau)}-(\mu+\gamma)I(\eta), \end{cases} \tag{1.3}$$

where $\eta = x+ct$ is the wave variable and c is the wave speed. A traveling wave solution of (1.1) is a pair of function $(S(\eta), I(\eta))$ satisfying (1.3) and the asymptotic boundary conditions

$$(S,I)(-\infty)=(1,0), \qquad (S,I)(\infty)=(S^*,I^*). \tag{1.4}$$

Zhou et al. [3] established the existence of a non-trivial super-critical travelling wave solution $(S(\eta), I(\eta))$ satisfying the asymptotic boundary condition (1.4). They also proved the non-existence of travelling wave solutions.

In contrast, little work has been done for the existence of critical travelling wave solutions for diffusive epidemic models [6,17,18]. Using the technique for dynamical system, Ding et al. [17] presented the existence of critical travelling wave solutions for a diffusive *SIS* epidemic model. By a limiting argument, Wang et al. [18] got the existence of critical traveling wave solutions for a diffusive epidemic model. Fu [6] showed the existence of critical travelling wave solutions of a diffusive *SIR* model with delay by constructing a pair of upper and lower solutions and applying the Schauder's fixed point theorem. Here we would like to point out that, due to the introduction of time delay, the applicability of the method in [17] to model (1) is not so obvious. We also note that the proof in [18] is problematic because the authors used the integrability of *I*-component in *SIR* model on the real line before they hadn't obtained the asymptotic boundary condition of the critical traveling wave. In [3], the existence of critical traveling wave solution of (1.1) was left as an open problem. In the present paper, inspired by [6], we intend to solve this problem.

Linearizing the second equation in (1.3) at $E^0=(1,0)$, one can obtain the corresponding characteristic equation

$$\Delta(\lambda,c)=d_2\lambda^2-c\lambda+\beta e^{-\lambda c\tau}-(\mu+\gamma)=0. \qquad (1.5)$$

For our purpose, we present the properties of (1.5) below, which have been given in [3].

Lemma 2.1. [3] Assume that $R_0>1$. Then there exists $c^*>0$ and $\lambda^*>0$, such that

$$\Delta(\lambda^*,c^*)=0 \quad \text{and} \quad \left.\frac{\partial\Delta(\lambda,c)}{\partial\lambda}\right|_{(\lambda^*,c^*)}:=\Delta_\lambda(\lambda^*,c^*)=0.$$

Moreover,

(i) if $0<c<c^*$, then $\Delta(\lambda,c)>0$ for any $\lambda\geq 0$;
(ii) if $c>c^*$, then $\Delta(\lambda,c)=0$ has two positive roots $\lambda_1(c)$ and $\lambda_2(c)$ with $0<\lambda_1(c)<\lambda^*<\lambda_2(c)$, such that

$$\Delta(\lambda,c)\begin{cases}>0 & for\ all\quad \lambda\in[0,\lambda_1(c))\cup(\lambda_2(c),\infty),\\ <0 & for\ all\quad \lambda\in(\lambda_1(c),\lambda_2(c)).\end{cases}$$

The main result of this paper is stated as follows.

Theorem 1.1. If $R_0>1$ and $c=c^*$, then system (1.1) admits a nontrivial, positive and bounded traveling wave solution satisfying the asymptotic boundary conditions (1.4).

Obviously, Theorem 1.1 supplements the result in [1]. From the epidemiological point of view, Theorem 1.1 together with the result in [3] determines whether the disease modelled by (1.1) can transmit and how fast it spreads if it can transmit. This can provide some insights into controlling the spread of the disease. Moreover, it can be seen that the critical wave speed c^* is defined as an implicit function of the model's parameters, which can help us to assess the control strategies. The method applied in this paper has prospect for deriving the existence of critical traveling wave solutions for the diffusive epidemic models in [7, 18].

The rest of this paper is organised as follows. Section 2-Section 4 are devoted to proving Theorem 1.1. In Section 2, we construct a pair of suitable upper and lower solutions of (1.3). In Section 3, we introduce a cone with this pair of upper and lower solutions and define a nonlinear operator. Then we establish the existence of critical travelling wave solution for (1.1) by Schauder's fixed point theorem. In Section 4, the asymptotic boundary of critical traveling wave solution is found by means of the squeeze theorem and Lyapunov functional method.

2. CONSTRUCTION OF UPPER AND LOWER SOLUTIONS

We introduce four continuous functions in R as follows:

$$\overline{S}(\eta)=1, \qquad \overline{I}(\eta)=\begin{cases}-L_1\eta e^{\lambda^*\eta}, & \eta\leq\eta_1,\\ \dfrac{\beta-\mu-\gamma}{\mu+\gamma}, & \eta>\eta_1,\end{cases}$$

$$\underline{S}(\eta)=\begin{cases}1-\dfrac{1}{\varepsilon}e^{\varepsilon\eta}, & \eta\leq\eta_2,\\ 0, & \eta>\eta_2,\end{cases} \qquad \underline{I}(\eta)=\begin{cases}\left[-L_1\eta-L_2(-\eta)^{\frac{1}{2}}\right]e^{\lambda^*\eta}, & \eta\leq\eta_3,\\ 0, & \eta>\eta_3,\end{cases}$$

where

$$L_1 := \frac{(\beta-\mu-\gamma)e\lambda^*}{\mu+\gamma}, \quad \eta_1 := -\frac{1}{\lambda^*}, \quad \eta_2 := \frac{1}{\varepsilon}\ln\varepsilon, \quad \eta_3 := -\left(\frac{L_2}{L_1}\right)^2,$$

ε, M and L_2 are three positive constants to be determined in the following lemmas.

Lemma 2.1. $\overline{S}(\eta)$ satisfies

$$d_1\overline{S}''(\eta) - c^*\overline{S}'(\eta) + \mu - \mu\overline{S}(\eta) - \frac{\beta\overline{S}(\eta)\underline{I}(\eta-c^*\tau)}{\overline{S}(\eta)+\underline{I}(\eta-c^*\tau)} \le 0 \tag{2.1}$$

for all $\eta \in R$.

Proof. Since $\overline{S}(\eta)=1$ in R, inequality (2.1) holds obviously.

Lemma 2.2. $\overline{I}(\eta)$ satisfies

$$d_2\overline{I}''(\eta) - c^*\overline{I}'(\eta) + \frac{\beta\overline{S}(\eta)\overline{I}(\eta-c^*\tau)}{\overline{S}(\eta)+\overline{I}(\eta-c^*\tau)} - (\mu+\gamma)\overline{I}(\eta) \le 0 \tag{2.2}$$

for all $\eta \ne \eta_1$.

Proof. If $\eta < \eta_1$, then $\overline{I}(\eta) = -L_1\eta e^{\lambda^*\eta}$. We have from Lemma 1.1 that

$$\begin{aligned}
& d_2\overline{I}''(\eta) - c^*\overline{I}'(\eta) + \frac{\beta\overline{S}(\eta)\overline{I}(\eta-c^*\tau)}{\overline{S}(\eta)+\overline{I}(\eta-c^*\tau)} - (\mu+\gamma)\overline{I}(\eta) \\
& \le d_2\overline{I}''(\eta) - c^*\overline{I}'(\eta) + \beta\overline{I}(\eta-c^*\tau) - (\mu+\gamma)\overline{I}(\eta) \\
& = -L_1\eta e^{\lambda^*\eta}\Delta(\lambda^*, c^*) \\
& = 0.
\end{aligned}$$

If $\eta > \eta_1$, then $\overline{I}(\eta) = \dfrac{\beta-\mu-\gamma}{\mu+\gamma}$. It follows from the monotonicity $\overline{I}$ that

$$\begin{aligned}
& d_2\overline{I}''(\eta) - c^*\overline{I}'(\eta) + \frac{\beta\overline{S}(\eta)\overline{I}(\eta-c^*\tau)}{\overline{S}(\eta)+\overline{I}(\eta-c^*\tau)} - (\mu+\gamma)\overline{I}(\eta) \\
& \le \frac{\beta\overline{S}(\eta)\overline{I}(\eta)}{\overline{S}(\eta)+\overline{I}(\eta)} - (\mu+\gamma)\overline{I}(\eta) \\
& = \left(\frac{\beta}{1+\frac{\beta-\mu-\gamma}{\mu+\gamma}} - \mu - \gamma\right)\overline{I}(\eta) \\
& = 0.
\end{aligned}$$

This completes the proof.

Lemma 2.3. Assume that $0<\varepsilon<\min\left\{\lambda^*,\dfrac{c^*}{d_1}\right\}$ is sufficiently small. Then $\underline{S}(\eta)$ satisfies

$$d_1\underline{S}''(\eta)-c^*\underline{S}'(\eta)+\mu-\mu\underline{S}(\eta)-\frac{\beta\underline{S}(\eta)\overline{I}(\eta-c^*\tau)}{\underline{S}(\eta)+\overline{I}(\eta-c^*\tau)}\geq 0 \tag{2.3}$$

for all $\eta\neq\eta_2$.

Proof. We choose $\varepsilon>0$ sufficiently small such that $\eta_2<\eta_1$. When $\eta>\eta_2$, $\underline{S}(\eta)=0$. Then (2.3) holds trivially. When $\eta<\eta_2$, $\underline{S}(\eta)=1-\dfrac{1}{\varepsilon}e^{\varepsilon\eta}$, $\overline{I}(\eta)=-L_1\eta e^{\lambda^*\eta}$. Then one can obtain for $\eta<\eta_2$ that

$$\begin{aligned}&d_1\underline{S}''(\eta)-c^*\underline{S}'(\eta)+\mu-\mu\underline{S}(\eta)-\frac{\beta\underline{S}(\eta)\overline{I}(\eta-c^*\tau)}{\underline{S}(\eta)+\overline{I}(\eta-c^*\tau)}\\&\geq d_1\underline{S}''(\eta)-c^*\underline{S}'(\eta)+\mu-\mu\underline{S}(\eta)-\beta\overline{I}(\eta)\\&=-d_1\varepsilon e^{\varepsilon\eta}+c^*e^{\varepsilon\eta}+\frac{\mu}{\varepsilon}e^{\varepsilon\eta}+L_1\beta e^{\lambda^*\eta}\\&=e^{\varepsilon\eta}(c^*-d_1\varepsilon)+e^{\varepsilon\eta}(\frac{\mu}{\varepsilon}+L_1\beta e^{(\lambda^*-\varepsilon)\eta}).\end{aligned} \tag{2.4}$$

Note that $\eta_2=\dfrac{1}{\varepsilon}\ln\varepsilon\to-\infty$ as $\varepsilon\to0^+$ and when $\eta<\eta_2$, $\dfrac{\mu}{\varepsilon}+L_1\beta e^{(\lambda^*-\varepsilon)\eta}\to\infty$ as $\varepsilon\to0^+$. Then we can use (2.4) to deduce that (2.2) holds for $0<\varepsilon<\min\{\lambda^*,\dfrac{c^*}{d_1}\}$ sufficiently small. The proof is completed.

Lemma 2.4. Let ε be fixed and satisfy Lemma 2.3. Then there exists a large enough constant $L_2>0$ such that the inequality

$$d_2\underline{I}''(\eta)-c^*\underline{I}'(\eta)+\frac{\beta\underline{S}(\eta)\underline{I}(\eta-c^*\tau)}{\underline{S}(\eta)+\underline{I}(\eta-c^*\tau)}-(\mu+\gamma)\underline{I}(\eta)\geq 0 \tag{2.5}$$

holds for all $\eta\neq\eta_3$.

Proof. We choose $L_2>0$ large enough such that $\eta_3<\eta_2$. When $\eta>\eta_3$, $\underline{I}(\eta)=0$. Then (2.5) holds trivially. When $\eta<\eta_3$, $\underline{S}(\eta)=1-\dfrac{1}{\varepsilon}e^{\varepsilon\eta}$, $\underline{I}(\eta)=\left[-L_1\eta-L_2(-\eta)^{\frac{1}{2}}\right]e^{\lambda^*\eta}$ and

$$\underline{I}(\eta-c^*\tau)=\left[-L_1(\eta-c^*\tau)-L_2(-\eta+c^*\tau)^{\frac{1}{2}}\right]e^{\lambda^*(\eta-c^*\tau)}.$$

By the Taylor's theorem, we have

$$(-\eta+c^*\tau)^{\frac{1}{2}}\leq(-\eta)^{\frac{1}{2}}+\frac{1}{2}c^*\tau(-\eta)^{-\frac{1}{2}}-\frac{1}{8}(c^*)^2\tau^2(-\eta)^{-\frac{3}{2}}+\frac{1}{16}(c^*\tau)^3(-\eta)^{-\frac{5}{2}}\quad\text{for }\eta<\eta_3.$$

Then we obtain from Lemma 1.1 that

$$
\begin{aligned}
& d_2\underline{I}''(\eta) - c^*\underline{I}'(\eta) + \frac{\beta\underline{S}(\eta)\underline{I}(\eta-c^*\tau)}{\underline{S}(\eta)+\underline{I}(\eta-c^*\tau)} - (\mu+\gamma)\underline{I}(\eta) \\
&= d_2\underline{I}''(\eta) - c^*\underline{I}'(\eta) + \beta\underline{I}(\eta-c^*\tau) - (\mu+\gamma)\underline{I}(\eta) - \frac{\beta\underline{I}^2(\eta-c^*\tau)}{\underline{S}(\eta)+\underline{I}(\eta-c^*\tau)} \\
&= \frac{1}{4}d_2L_2(-\eta)^{-\frac{3}{2}}e^{\lambda^*\eta} + 2d_2\lambda^*\left[-L_1+\frac{1}{2}L_2(-\eta)^{-\frac{1}{2}}\right]e^{\lambda^*\eta} + d_2(\lambda^*)^2\left[-L_1\eta - L_2(-\eta)^{\frac{1}{2}}\right]e^{\lambda^*\eta} \\
&\quad - c^*\left[-L_1+\frac{1}{2}L_2(-\eta)^{-\frac{1}{2}}\right]e^{\lambda^*\eta} - c^*\lambda^*\left[-L_1\eta - L_2(-\eta)^{\frac{1}{2}}\right]e^{\lambda^*\eta} \\
&\quad + \beta\left[-L_1(\eta-c^*\tau) - L_2(-\eta+c^*\tau)^{\frac{1}{2}}\right]e^{\lambda^*(\eta-c^*\tau)} - (\mu+\gamma)\left[-L_1\eta - L_2(-\eta)^{\frac{1}{2}}\right]e^{\lambda^*\eta} \\
&\quad - \frac{\beta\underline{I}^2(\eta-c^*\tau)}{\underline{S}(\eta)+\underline{I}(\eta-c^*\tau)} \\
&\geq \left[-L_1\eta - L_2(-\eta)^{\frac{1}{2}}\right]e^{\lambda^*\eta}\Delta(\lambda^*,c^*) + \left[-L_1+\frac{1}{2}L_2(-\eta)^{-\frac{1}{2}}\right]e^{\lambda^*\eta}\Delta_\lambda(\lambda^*,c^*) \\
&\quad - \beta L_2\left[-\frac{(c^*\tau)^2}{8}(-\eta)^{-\frac{3}{2}} + \frac{(c^*\tau)^3}{16}(-\eta)^{-\frac{5}{2}}\right]e^{\lambda^*(\eta-c^*\tau)} - \frac{\beta L_1^2(\eta-c^*\tau)^2e^{2\lambda^*(\eta-c^*\tau)}}{1-\frac{1}{\varepsilon}e^{\varepsilon\eta}} \\
&\geq \beta L_2(-\eta)^{-\frac{3}{2}}e^{\lambda^*(\eta-c^*\tau)}\frac{(c^*\tau)^2}{16}\left(1+\frac{c^*\tau}{\eta}\right) + \frac{1}{16}\beta L_2(c^*\tau)^2(-\eta)^{-\frac{3}{2}}e^{\lambda^*(\eta-c^*\tau)} \\
&\quad - \frac{\beta L_1^2(\eta-c^*\tau)^2e^{2\lambda^*(\eta-c^*\tau)}}{1-\frac{1}{\varepsilon}e^{\varepsilon\eta}} \qquad (2.6) \\
&= \beta L_2(-\eta)^{-\frac{3}{2}}e^{\lambda^*(\eta-c^*\tau)}\frac{(c^*\tau)^2}{16}\left(1+\frac{c^*\tau}{\eta}\right) \\
&\quad + \frac{\beta(-\eta)^{-\frac{3}{2}}e^{\lambda^*(\eta-c^*\tau)}\left[\frac{(c^*\tau)^2}{16}L_2\left(1-\frac{1}{\varepsilon}e^{\varepsilon\eta}\right) - L_1^2(-\eta)^{\frac{3}{2}}(\eta-c^*\tau)^2e^{\lambda^*(\eta-c^*\tau)}\right]}{1-\frac{1}{\varepsilon}e^{\varepsilon\eta}}
\end{aligned}
$$

Note that $\eta_3 = -\left(\frac{L_2}{L_1}\right)^2 \to -\infty$ as $L_2 \to \infty$, then we get that, for $\eta < \eta_3$, $1+\frac{c^*\tau}{\eta} \to 1$, $\frac{(c^*\tau)^2}{16}L_2\left(1-\frac{1}{\varepsilon}e^{\varepsilon\eta}\right) \to \infty$ and $L_1^2(-\eta)^{\frac{3}{2}}(\eta-c^*\tau)^2e^{\lambda^*(\eta-c^*\tau)} \to 0$ as $L_2 \to \infty$. Therefore, we conclude that (2.6) holds for $L_2 > 0$ large enough. The proof is finished.

3. APPLICATION OF SCHAUDER'S FIXED POINT THEOREM

System (1.2) can be rewritten in an equivalent form:

$$\begin{cases} d_1 S''(\eta) - c^* S'(\eta) - \alpha_1 S(\eta) + \alpha_1 S(\eta) + \mu - \mu S(\eta) - \dfrac{\beta S(\eta) I(\eta - c^*\tau)}{S(\eta) + I(\eta - c^*\tau)} = 0, \\ d_2 I''(\eta) - c^* I'(\eta) - \alpha_2 I(\eta) + \alpha_2 I(\eta) + \dfrac{\beta S(\eta) I(\eta - c^*\tau)}{S(\eta) + I(\eta - c^*\tau)} - (\mu + \gamma) I(\eta) = 0. \end{cases}$$

For our purpose, here the constants α_1 and α_2 are chosen to satisfy $\alpha_1 > \mu + \beta$ and $\alpha_2 > \mu + \gamma$, respectively.

Let $\lambda_i^{\pm}$ be the roots of equations $d_i \lambda^2 - c^* \lambda - \alpha_i = 0 \ \ (i = 1,2)$, then one has

$$\lambda_i^{\pm} = \frac{c^* \pm \sqrt{(c^*)^2 + 4 d_i \alpha_i}}{2 d_i}, \qquad i = 1,2 .$$

Introduce a set

$$\Gamma := \left\{ (S, I)(\eta) \in B_\mu(R, R^2) \,\middle|\, \underline{S}(\eta) \le S(\eta) \le \overline{S}(\eta),\ \underline{I}(\eta) \le I(\eta) \le \overline{I}(\eta) \right\},$$

Where

$$B_\mu(R, R^2) = \max\left\{ \Phi = (\phi(\cdot), \varphi(\cdot)) \in C(R, R^2) \,\middle|\, \sup_{\eta \in R} |\phi(\eta)| e^{-\mu|\eta|} < \infty, \sup_{\eta \in R} |\varphi(\eta)| e^{-\mu|\eta|} < \infty \right\}$$

equipped with the norm

$$|\Phi|_\mu := \max\left\{ \sup_{\eta \in R} |\phi(\eta)| e^{-\mu|\eta|},\ \sup_{\eta \in R} |\varphi(\eta)| e^{-\mu|\eta|} \right\}.$$

Here the constant μ satisfies $0 < \mu < \min\{-\lambda_1^-, -\lambda_2^-\}$. Then $B_\mu(R, R^2)$ is a Banach space with the norm $|\cdot|_\mu$.

Define a function $g : \Gamma \to C(R)$

$$g(S, I)(\eta) = \begin{cases} \dfrac{\beta S(\eta) I(\eta - c^*\tau)}{S(\eta) + I(\eta - c^*\tau)}, & S(\eta) I(\eta - c^*\tau) \ne 0, \\ 0, & S(\eta) I(\eta - c^*\tau) = 0 \end{cases}$$

and an operator: $F = (F_1(S, I)(\eta), F_2(S, I)(\eta)) : \Gamma \to C(R, R^2)$, where

$$F_i(S, I)(\eta) = \frac{1}{\Lambda_i} \left(\int_{-\infty}^{\eta} e^{\lambda_i^-(\eta - x)} h_i(S, I)(x) dx + \int_{\eta}^{\infty} e^{\lambda_i^+(\eta - x)} h_i(S, I)(x) dx \right),$$

$$\Lambda_i = d_i(\lambda_i^+ - \lambda_i^-) = \sqrt{(c^*)^2 + 4 d_i \alpha_i}, \quad i = 1,2,$$

$$h_1(S, I)(\eta) = \alpha_1 S(\eta) + \mu - \mu S(\eta) - g(S, I)(\eta)$$

and

$$h_2(S, I)(\eta) = (\alpha_2 - \mu - \gamma) I(\eta) + g(S, I)(\eta).$$

Lemma 3.1. The operator F maps Γ into Γ.

Proof. For $(S(\eta), I(\eta)) \in \Gamma$, it is sufficient to prove that

$$\underline{S}(\eta) \leq S(\eta) \leq \overline{S}(\eta), \ \underline{I}(\eta) \leq I(\eta) \leq \overline{I}(\eta), \quad \eta \in R.$$

Let $(S(\eta), I(\eta)) \in \Gamma$, then we deduce from Lemma 2.1 that

$$\begin{aligned}
F_1(S,I)(\eta) &= \frac{1}{\Lambda_1}\Big(\int_{-\infty}^{\eta} e^{\lambda_1^-(\eta-x)}(\alpha_1 S(x) + \mu - \mu S(x) - g(S,I)(x))dx \\
&\quad + \int_{\eta}^{\infty} e^{\lambda_1^+(\eta-x)}(\alpha_1 S(x) + \mu - \mu S(x) - g(S,I)(x))dx\Big) \\
&\leq \frac{1}{\Lambda_1}\Big(\int_{-\infty}^{\eta} e^{\lambda_1^-(\eta-x)}(\alpha_1 \overline{S}(x) + \mu - \mu \overline{S}(x) - g(\overline{S},\underline{I})(x))dx \\
&\quad + \int_{\eta}^{\infty} e^{\lambda_1^+(\eta-x)}(\alpha_1 \overline{S}(x) + \mu - \mu \overline{S}(x) - g(\overline{S},\underline{I})(x))dx\Big) \\
&\leq \frac{1}{\Lambda_1}\Big(\int_{-\infty}^{\eta} e^{\lambda_1^-(\eta-x)}(\alpha_1 \overline{S}(x) + c^* \overline{S}'(x) - d_1 \overline{S}''(x))dx \\
&\quad + \int_{\eta}^{\infty} e^{\lambda_1^+(\eta-x)}(\alpha_1 \overline{S}(x) + c^* \overline{S}'(x) - d_1 \overline{S}''(x))dx\Big) \\
&\leq \frac{1}{\Lambda_1}\Big(\int_{-\infty}^{\eta} e^{\lambda_1^-(\eta-x)}\alpha_1 dx + \int_{\eta}^{\infty} e^{\lambda_1^+(\eta-x)}\alpha_1 dx\Big) \\
&= \frac{d_1}{\Lambda_1}\left(\frac{1}{\lambda_1^+} - \frac{1}{\lambda_1^-}\right) \\
&= 1.
\end{aligned}$$

When $\eta < \eta_2$, $\underline{S}(\eta) = 1 - \frac{1}{\varepsilon} e^{\varepsilon\eta}$. From Lemma 2.3 we obtain that

$$\begin{aligned}
F_1(S,I)(\eta) &= \frac{1}{\Lambda_1}\Big(\int_{-\infty}^{\eta} e^{\lambda_1^-(\eta-x)}(\alpha_1 S(x) + \mu - \mu S(x) - g(S,I)(x))dx \\
&\quad + \int_{\eta}^{\infty} e^{\lambda_1^+(\eta-x)}(\alpha_1 S(x) + \mu - \mu S(x) - g(S,I)(x))dx\Big) \\
&\geq \frac{1}{\Lambda_1}\Big(\int_{-\infty}^{\eta} e^{\lambda_1^-(\eta-x)}(\alpha_1 \underline{S}(x) + \mu - \mu \underline{S}(x) - g(\underline{S},\overline{I})(x))dx \\
&\quad + \int_{\eta}^{\infty} e^{\lambda_1^+(\eta-x)}(\alpha_1 \underline{S}(x) + \mu - \mu \underline{S}(x) - g(\underline{S},\overline{I})(x))dx\Big) \\
&\geq \frac{1}{\Lambda_1}\Big(\int_{-\infty}^{\eta} e^{\lambda_1^-(\eta-x)}(\alpha_1 \underline{S}(x) + c^* \underline{S}'(x) - d_1 \underline{S}''(x))dx \\
&\quad + \int_{\eta}^{\infty} e^{\lambda_1^+(\eta-x)}(\alpha_1 \underline{S}(x) + c^* \underline{S}'(x) - d_1 \underline{S}''(x))dx\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Lambda_1}\Big(\int_{-\infty}^{\eta} e^{\lambda_1^-(\eta-x)}\Big(\alpha_1 + \tfrac{1}{\varepsilon}(d_1\varepsilon^2 - c^*\varepsilon - \alpha_1)e^{\varepsilon x}\Big)dx \\
&\quad + \int_{\eta}^{\eta_2} e^{\lambda_1^+(\eta-x)}\Big(\alpha_1 + \tfrac{1}{\varepsilon}(d_1\varepsilon^2 - c^*\varepsilon - \alpha_1)e^{\varepsilon x}\Big)dx\Big) \\
&= 1 - \frac{1}{\varepsilon}e^{\varepsilon\eta} + \frac{d_1\varepsilon}{\Lambda_1}e^{\lambda_1^+(\eta-\eta_2)} \quad for\ \ \eta < \eta_2.
\end{aligned} \tag{3.1}
$$

When $\eta > \eta_2$, one can have that

$$
\begin{aligned}
F_1(S,I)(\eta) &\geq \frac{1}{\Lambda_1}\Big(\int_{-\infty}^{\eta} e^{\lambda_1^-(\eta-x)}(\alpha_1-\beta)S(x)dx + \int_{\eta}^{\infty} e^{\lambda_1^+(\eta-x)}(\alpha_1-\beta)S(x)dx\Big) \\
&\geq \frac{1}{\Lambda_1}\Big(\int_{-\infty}^{\eta} e^{\lambda_1^-(\eta-x)}(\alpha_1-\beta)\underline{S}(x)dx + \int_{\eta}^{\infty} e^{\lambda_1^+(\eta-x)}(\alpha_1-\beta)\underline{S}(x)dx\Big) \\
&\geq 0.
\end{aligned} \tag{3.2}
$$

From (3.1), (3.2) and the continuity of $F_1(S,I)(\eta)$ and $\underline{S}(\eta)$ in R, we conclude that

$F_1(S,I)(\eta) \geq \underline{S}(\eta)$ for $\eta \in R$.

When $\eta < \eta_1$, $\overline{I}(\eta) = -L_1\eta e^{\lambda^*\eta}$. In view of Lemma 2.2, we obtain that

$$
\begin{aligned}
F_2(S,I)(\eta) &= \frac{1}{\Lambda_2}\Big(\int_{-\infty}^{\eta} e^{\lambda_2^-(\eta-x)}\big(\alpha_2 I(x) + g(S,I)(x) - (\mu+\gamma)I(x)\big)dx \\
&\quad + \int_{\eta}^{\infty} e^{\lambda_2^+(\eta-x)}\big(\alpha_2 I(x) + g(S,I)(x) - (\mu+\gamma)I(x)\big)dx\Big) \\
&\leq \frac{1}{\Lambda_2}\Big(\int_{-\infty}^{\eta} e^{\lambda_2^-(\eta-x)}\big(\alpha_2 \overline{I}(x) + g(\overline{S},\overline{I})(x) - (\mu+\gamma)\overline{I}(x)\big)dx \\
&\quad + \int_{\eta}^{\eta_1} e^{\lambda_2^+(\eta-x)}\big(\alpha_2 \overline{I}(x) + g(\overline{S},\overline{I})(x) - (\mu+\gamma)\overline{I}(x)\big)dx \\
&\quad + \int_{\eta_1}^{\infty} e^{\lambda_2^+(\eta-x)}\big(\alpha_2 \overline{I}(x) + g(\overline{S},\overline{I})(x) - (\mu+\gamma)\overline{I}(x)\big)dx\Big) \\
&\leq \frac{1}{\Lambda_2}\Big(\int_{-\infty}^{\eta} e^{\lambda_2^-(\eta-x)}(\alpha_2\overline{I}(x) + c^*\overline{I}'(x) - d_2\overline{I}''(x))dx \\
&\quad + \int_{\eta}^{\eta_1} e^{\lambda_2^+(\eta-x)}(\alpha_2\overline{I}(x) + c^*\overline{I}'(x) - d_2\overline{I}''(x))dx \\
&\quad + \int_{\eta_1}^{\infty} e^{\lambda_2^+(\eta-x)}(\alpha_2\overline{I}(x) + c^*\overline{I}'(x) - d_2\overline{I}''(x))dx\Big) \\
&= \frac{1}{\Lambda_2}\Big(\int_{-\infty}^{\eta} e^{\lambda_2^-(\eta-x)}\big(d_2(\lambda^*)^2 - c^*\lambda^* - \alpha_2\big)L_1 x e^{\lambda^* x}dx \\
&\quad + \int_{\eta}^{\eta_1} e^{\lambda_2^+(\eta-x)}\big(d_2(\lambda^*)^2 - c^*\lambda^* - \alpha_2\big)L_1 x e^{\lambda^* x}dx \\
&\quad + \int_{\eta_1}^{\infty} e^{\lambda_2^+(\eta-x)}\frac{\alpha_2(\beta-\mu-\gamma)}{\mu+\gamma}dx\Big)
\end{aligned} \tag{3.3}
$$

$$= \frac{1}{\Lambda_2} L_1 d_2 (\lambda_2^- - \lambda_2^+) \eta e^{\lambda^* \eta}$$
$$= -L_1 \eta e^{\lambda^* \eta} \quad for \quad \eta < \eta_1.$$

If $\eta > \eta_1$, then $\overline{I}(\eta) = \dfrac{\beta - \mu - \gamma}{\mu + \gamma}$. It follows from Lemma 2.2 that

$$\begin{aligned} F_2(S,I)(\eta) \le & \frac{1}{\Lambda_2}\Big(\int_{-\infty}^{\eta_1} e^{\lambda_2^-(\eta-x)}\big(\alpha_2 \overline{I}(x) + g(\overline{S},\overline{I})(x) - (\mu+\gamma)\overline{I}(x)\big)dx \\ & + \int_{\eta_1}^{\eta} e^{\lambda_2^-(\eta-x)}\big(\alpha_2 \overline{I}(x) + g(\overline{S},\overline{I})(x) - (\mu+\gamma)\overline{I}(x)\big)dx \\ & + \int_{\eta}^{\infty} e^{\lambda_2^+(\eta-x)}\big(\alpha_2 \overline{I}(x) + g(\overline{S},\overline{I})(x) - (\mu+\gamma)\overline{I}(x)\big)dx\Big) \\ \le & \frac{1}{\Lambda_2}\Big(\int_{-\infty}^{\eta_1} e^{\lambda_2^-(\eta-x)}(\alpha_2 \overline{I}(x) + c^* \overline{I}'(x) - d_2 \overline{I}''(x))dx \\ & + \int_{\eta_1}^{\eta} e^{\lambda_2^-(\eta-x)}(\alpha_2 \overline{I}(x) + c^* \overline{I}'(x) - d_2 \overline{I}''(x))dx \\ & + \int_{\eta}^{\infty} e^{\lambda_2^+(\eta-x)}(\alpha_2 \overline{I}(x) + c^* \overline{I}'(x) - d_2 \overline{I}''(x))dx\Big) \\ = & \frac{1}{\Lambda_2}\Big(\int_{-\infty}^{\eta_1} e^{\lambda_2^-(\eta-x)}\big(d_2(\lambda^*)^2 - c^*\lambda^* - \alpha_2\big)L_1 x e^{\lambda^* x}dx \\ & + \int_{\eta_1}^{\eta} e^{\lambda_2^-(\eta-x)} \frac{\alpha_2(\beta-\mu-\gamma)}{\mu+\gamma}dx + \int_{\eta}^{\infty} e^{\lambda_2^+(\eta-x)} \frac{\alpha_2(\beta-\mu-\gamma)}{\mu+\gamma}dx\Big) \\ = & \frac{\beta-\mu-\gamma}{\mu+\gamma} \quad for \quad \eta > \eta_1. \end{aligned} \tag{3.4}$$

By (3.3), (3.4) and the continuity of $F_2(S,I)(\eta)$ and $\overline{I}(\eta)$ in R, we obtain that

$$F_2(S,I)(\eta) \le \overline{I}(\eta) \quad \text{for} \quad \eta \in R.$$

When $\eta < \eta_3$, $\underline{I}(\eta) = [-L_1 \eta - L_2(-\eta)^{\frac{1}{2}}]e^{\lambda^* \eta}$. It follows from Lemma 2.4 that

$$\begin{aligned} F_2(S,I)(\eta) = & \frac{1}{\Lambda_2}\Big(\int_{-\infty}^{\eta} e^{\lambda_2^-(\eta-x)}\big(\alpha_2 I(x) + g(S,I)(x) - (\mu+\gamma)I(x)\big)dx \\ & + \int_{\eta}^{\infty} e^{\lambda_2^+(\eta-x)}\big(\alpha_2 I(x) + g(S,I)(x) - (\mu+\gamma)I(x)\big)dx\Big) \\ \ge & \frac{1}{\Lambda_2}\Big(\int_{-\infty}^{\eta} e^{\lambda_2^-(\eta-x)}\big(\alpha_2 \underline{I}(x) + g(\underline{S},\underline{I})(x) - (\mu+\gamma)\underline{I}(x)\big)dx \end{aligned}$$

$$
\begin{aligned}
&+\int_{\eta}^{\eta_3} e^{\lambda_2^+(\eta-x)}\big(\alpha_2\underline{I}(x)+g(\underline{S},\underline{I})(x)-(\mu+\gamma)\underline{I}(x)\big)dx\Big)\\
&\geq \frac{1}{\Lambda_2}\Big(\int_{-\infty}^{\eta} e^{\lambda_2^-(\eta-x)}(\alpha_2\underline{I}(x)+c^*\underline{I}'(x)-d_2\underline{I}''(x))dx\\
&+\int_{\eta}^{\eta_3} e^{\lambda_2^+(\eta-x)}(\alpha_2\underline{I}(x)+c^*\underline{I}'(x)-d_2\underline{I}''(x))dx\Big)\\
&=\frac{1}{\Lambda_2}\Big(\int_{-\infty}^{\eta} e^{\lambda_2^-(\eta-x)}\Big((d_2(\lambda^*)^2-c^*\lambda^*-\alpha_2)L_1xe^{\lambda^*x}\\
&+L_2(d_2(\lambda^*)^2-c^*\lambda^*-\alpha_2)(-x)^{\frac{1}{2}}e^{\lambda^*x}-\frac{1}{4}d_2L_2(-x)^{-\frac{3}{2}}e^{\lambda^*x}\Big)dx\\
&+\int_{\eta}^{\eta_3} e^{\lambda_2^+(\eta-x)}\Big((d_2(\lambda^*)^2-c^*\lambda^*-\alpha_2)L_1xe^{\lambda^*x}\\
&+L_2(d_2(\lambda^*)^2-c^*\lambda^*-\alpha_2)(-x)^{\frac{1}{2}}e^{\lambda^*x}-\frac{1}{4}d_2L_2(-x)^{-\frac{3}{2}}e^{\lambda^*x}\Big)dx\Big)\\
&=\left[-L_1\eta-L_2(-\eta)^{\frac{1}{2}}\right]e^{\lambda^*\eta}+\frac{L_1d_2}{2\Lambda_2}e^{\lambda^*\eta_3}e^{\lambda_2^+(\eta-\eta_3)}\\
&\geq\left[-L_1\eta-L_2(-\eta)^{\frac{1}{2}}\right]e^{\lambda^*\eta}\quad for\quad \eta<\eta_3.
\end{aligned}
\tag{3.5}
$$

When $\eta>\eta_3$, $\underline{I}(\eta)=0$. We get that

$$
\begin{aligned}
F_2(S,I)(\eta) &\geq \frac{1}{\Lambda_2}\int_{-\infty}^{\eta_3} e^{\lambda_2^-(\eta-x)}(\alpha_2\underline{I}(x)-(\mu+\gamma)\underline{I}(x))dx\\
&\geq 0\qquad for\quad \eta>\eta_3,
\end{aligned}
\tag{3.6}
$$

which together with (3.5) and the continuity of $F_2(S,I)(\eta)$ and $\underline{I}(\eta)$ in R imply that

$$F_2(S,I)(\eta)\geq \underline{I}(\eta)\quad \text{for}\quad \eta\in R.$$

The claim of this lemma is shown.

Lemma 3.2. The operator F is continuous and compact with respect to the norm $|\cdot|_\mu$ in $B_\mu(R,R^2)$.

Proof. The proof is similar to the case $c>c^*$ in [1]. So we omit the details here for brevity.

4. EXISTENCE AND ASYMPTOTIC BOUNDARY OF TRAVELING WAVE SOLUTION WITH CRITICAL SPEED

Proposition 4.1. Suppose that $R_0>1$ and $c=c^*$, then (1.1) admits a non-trivial positive traveling wave solution $(S(x+c^*\tau),I(x+c^*\tau))$ satisfying the asymptotic boundary conditions (1.4). Moreover, $I(\eta)=O(-\eta e^{\lambda^*\eta})$ as $\eta\to-\infty$.

Proof. Applying Lemma 3.1, Lemma 3.2 and Schauder's fixed point theorem, we conclude that F has a fixed point $(S_*(\eta), I_*(\eta)) \in \Gamma$, that is, $S_*(\eta) = F_1(S_*, I_*)(\eta)$, $I_*(\eta) = F_2(S_*, I_*)(\eta)$, which is also a traveling wave solution of (1.1) satisfying

$$\underline{S}(\eta) \le S_*(\eta) \le \overline{S}(\eta) \quad \text{and} \quad \underline{I}(\eta) \le I_*(\eta) \le \overline{I}(\eta) \quad \text{for } \eta \in R. \tag{4.1}$$

From (4.1) and squeeze theorem, we obtain that

$$S_*(-\infty) = 1,\; I_*(-\infty) = 0 \text{ and } I_*(\eta) = O(-\eta e^{\lambda^* \eta}) \quad \text{as} \quad \eta \to -\infty .$$

Now we prove $0 < S_*(\eta) < 1$ and $0 < I_*(\eta) < \frac{\beta - \mu - \gamma}{\mu + \gamma}$ for $\eta \in R$. Assume by the way of contradiction that $S_*(\tilde{\eta}) = 0$ for some $\tilde{\eta} \in R$. Then there exist two constants $m_1, m_2 \in R$ such that $m_1 < \eta_2 \le m_2$ and $\tilde{\eta} \in (m_1, m_2)$. This implies that $S(\eta)$ attains its minimum in (m_1, m_2). From the first equation in (1.2) we get

$$-d_1 S_*''(\eta) + c^* S_*'(\eta) + \beta S_*(\eta) = \mu - \mu S_*(\eta) + \beta S_*(\eta) - \frac{\beta S_*(\eta) I_*(\eta - c^*\tau)}{S_*(\eta) + I_*(\eta - c^*\tau)} \ge 0$$

for $\eta \in (m_1, m_2)$. Then one can use the strong maximum principle to get that

$$S_*(\eta) \equiv 0 \quad \text{for } \eta \in (m_1, m_2),$$

which contradicts the fact that $S_*(\eta) \ge \underline{S}(\eta) > 0$ for $\eta \in [m_1, \eta_2)$. Hence we have

$$S_*(\eta) > 0 \quad \text{for} \quad \eta \in R .$$

Analogously, we have

$$I_*(\eta) > 0 \quad \text{for} \quad \eta \in R .$$

To prove $S_*(\eta) < 1$ for $\eta \in R$, we assume that $S_*(\hat{\eta}) = 1$ for some $\hat{\eta} \in R$. Then $S_*'(\hat{\eta}) = 0$, $S_*''(\hat{\eta}) \le 0$. This contradicts the first equation in (1.2) at the point $\eta = \hat{\eta}$. Thus

$$S_*(\eta) < 1 \quad \text{for} \quad \eta \in R .$$

Similarly, we have

$$0 < I_*(\eta) < \frac{\beta - \mu - \gamma}{\mu + \gamma} \quad \text{for} \quad \eta \in R.$$

In the following, we intend to find the asymptotic boundary of (S_*, I_*) at plus infinity. To the end, we consider the Lyapunov function:

$$\begin{aligned} V(S,I)(\eta) = {} & c^* \left(S(\eta) - S_* - \int_{S^*}^{S(\eta)} \frac{h(S^*, I^*)}{h(y, I^*)} dy + I(\eta) - I^* - I^* \ln \frac{I(\eta)}{I^*} \right) \\ & + \beta h(S^*, I^*) \int_0^{c^*\tau} \left(\frac{I(\eta - \theta)}{I^*} - 1 - \ln \frac{I(\eta - \theta)}{I^*} \right) d\theta \\ & + d_1 S'(\eta) \left(\frac{h(S^*, I^*)}{h(S(\eta), I^*)} - 1 \right) + d_2 I'(\eta) \left(\frac{I^*}{I(\eta)} - 1 \right), \end{aligned}$$

where $h(S,I)=\dfrac{SI}{S+I}$. By a similar argument to that in [3, Lemma 3.1], one can see that V is well-defined and bounded from blow. Following the discussion on the case $c>c^*$ in [3], one can obtain that $\dfrac{dV(S,I)(\eta)}{d\eta}\le 0$ and $\dfrac{dV(S,I)(\eta)}{d\eta}=0$ if and only if $S(\eta)\equiv S^*$, $I(\eta)\equiv I^*$, $S'(\eta)\equiv 0$ and $I'(\eta)\equiv 0$.

Now we choose an increasing sequence $\{\eta_n\}$ satisfying $\lim_{n\to\infty}\eta_n=\infty$ and denote

$$\{S_{*,n}(\eta)\}_{n=1}^{\infty}=\{S_*(\eta+\eta_n)\}_{n=1}^{\infty},\quad \{I_{*,n}(\eta)\}_{n=1}^{\infty}=\{I_*(\eta+\eta_n)\}_{n=1}^{\infty},$$

$$\{S'_{*,n}(\eta)\}_{n=1}^{\infty}=\{S_*'(\eta+\eta_n)\}_{n=1}^{\infty},\quad \{I'_{*,n}(\eta)\}_{n=1}^{\infty}=\{I_*'(\eta+\eta_n)\}_{n=1}^{\infty}.$$

Since $\{S_{*,n}(\eta)\}_{n=1}^{\infty}$, $\{I_{*,n}(\eta)\}_{n=1}^{\infty}$, $\{S'_{*,n}(\eta)\}_{n=1}^{\infty}$ and $\{I'_{*,n}(\eta)\}_{n=1}^{\infty}$ are uniformly bounded in R, we infer that there exists a subsequence of functions, still denoted by $S_{*,n}, I_{*,n}, S'_{*,n}$ and $I'_{*,n}$, respectively, such that

$$\lim_{n\to\infty}S_{*,n}(\eta)=\widetilde{S}_*(\eta),\quad \lim_{n\to\infty}I_{*,n}(\eta)=\widetilde{I}_*(\eta),$$

$$\lim_{n\to\infty}S'_{*,n}(\eta)=\widetilde{S}'_*(\eta),\quad \lim_{n\to\infty}I'_{*,n}(\eta)=\widetilde{I}'_*(\eta).$$

Then Lebesgue dominated convergence theorem gives that

$$\lim_{n\to\infty}V(S_{*,n},I_{*,n})(\eta)=V(\widetilde{S}_*,\widetilde{I}_*)(\eta). \tag{4.2}$$

Note that $V(S_*,I_*)(\eta)$ is non-increasing and bounded from below, then we have that for $n\in N$, there existes a constant $\widetilde{C}$ such that

$$\widetilde{C}\le V(S_{*,n},I_{*,n})(\eta)=V(S_*,I_*)(\eta+\eta_n)\le V(S_*,I_*)(\eta),$$

which indicates that

$$\lim_{n\to\infty}V(S_{*,n},I_{*,n})(\eta)=\lim_{\eta+\eta_n\to\infty}V(S_*,I_*)(\eta+\eta_n):=V_0 \tag{4.3}$$

exists for all $\eta\in R$, where V_0 is a constant. By (4.2) and (4.3) we get that $V(\widetilde{S}_*,\widetilde{I}_*)(\eta)=V_0$, which leads to

$$\frac{dV(\widetilde{S}_*,\widetilde{I}_*)(\eta)}{d\eta}=0 .$$

This implies that $(S_*,I_*)(\infty)=(S^*,I^*)$. This proof is completed.

ACKNOWLEDGEMENTS

The authors are grateful to the handling editor and the anonymous referees for their careful reading and critical comments which have helped them to improve the presentation of this work significantly.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

1. Murray JD. Mathematical Biology, I and II, third edn, Springer; 2002.
2. Abualrub MS. An analysis to the traveling wave solution of a diffusive model of an epidemic. Tamkang J. Math. 1998;29:65-68.
3. Zhou K, Han, M, Wang Q. Traveling wave solutions for a delayed diffusive SIR epidemic model with nonlinear incidence rate and external supplies. Math. Method. Appl. Sci. 2016;40:2772-2783.
4. Yang J, Liang S, Zhang Y. Travelling waves of a delayed SIR epidemic model with nonlinear incidence rate and spatial diffusion. PLoS One. 2011;6:e21128.
5. Gan Q, Xu R, Yang P. Travelling waves of a delayed SIRS epidemic model with spatial diffusion. Nonlinear Anal. Real. 2011;12:52-68.
6. Fu S. Traveling waves for a diffusive SIR model with delay. J. Math. Anal. Appl. 2016;435:20-37.
7. Lotfi, EM, Hattaf K, Yousf N. Global dynamics and traveling waves of a delayed diffusive epidemic model with specific nonlinear incidence rate. Brit. J. Math. Comput. Sci. 2017;20:1-17.
8. Du Y, Xu R. Traveling waves of an SIRS epidemic model with spatial diffusion and time delay. J. Appl. Math. Inform. 2012;30:635-646.
9. Xu Z, Chen D. An SIS epidemic model with diffusion. Appl. Math. J. Chinese Univ. 2017;32:127-146.
10. Ducrot A, Langlais M, Magal P. Qualitative analysis and travelling wave solutions for the SI model with vertical transmission. Commum. Pure Appl. Anal. 2012;11:97-113.
11. Zhu C, Li W, Yang F. Traveling waves of a reaction-diffusion SIRQ epidemic model with relapse. J. Appl. Anal. Comput. 2017;7:147-171.
12. Weng P, Zhao XQ. Spreading speed and traveling waves for a multi-type SIS epidemics model. J Differential Equations. 2006;229:270-296.
13. Zhang T, Wang W. Existence of traveling wave solutions for influenza model with treatment. J Math Anal Appl. 2014;419: 469-495.
14. Hsu C, Yang T. Existence, uniqueness, monotonicity and asymptotic behaviour of travelling waves for epidemic models. Nonlinearity. 2013;26:121-139.
15. Wang Z, Wu J, Liu R. Traveling waves of the spread of avian influenza, Proc. Amer. Math. Soc. 2012;140:3931–3946.
16. Ai S, Albashaireh R. Traveling waves in spatial SIRS models. J. Dyn. Diff. Equat. 2014;26:143-164.
17. Ding W, Huang W, Kansakar S. Traveling wave solutions for a diffusive SIS epidemic model. Discrete Contin. Dyn. Syst. B 2013;18:1291-1304.
18. Wang Z, Nie H, Wu J. Existence and uniqueness of traveling waves for a reaction-diffusion model with general response functions, J. Math. Anal. Appl. 2017;450:406-426.
19. Zhan F, Huo G, Liu Q, Sun G, Jin Z. Existence of traveling waves in nonlinear SI epidemic models. J. Biol. Syst. 2009;17: 643-657.

Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sciencedomain.org/review-history/27304