British Journal of Mathematics & Computer Science
3(2): 164-179, 2013

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

On Removable Sets for Generated Elliptic Equations

T.S. Gadjiev[*1] **and N.Q.Bayramova**[1]

[1] *Institute of Mathematics and Mechanics of NAS of Azerbaijan,*
B.Vahabzade 9, AZ 1141, Baku, Azerbaijan

Research Article

Received: 31 October 2012
Accepted: 06 February 2013
Published: 29 March 2013

Abstract

In the paper the necessary and sufficient condition of compact removability is obtained

Keywords: elliptic, equation, removability, necessary, sufficient condition, classes of bounded functions
2010 Mathematics Subject Classification: 35J15

1 Introduction

The questions of compact removability for Laplace equation is studied by [1]. The uniform elliptic equation of the second order of divergent structure is studied by [2]. The compact removability for elliptic and parabolic equations of nondivergent structure is considered by [3], [4]. The removability condition of compact in the space of continuous functions are constructed in the papers of [5], [6]. The different questions of qualitative properties of solutions of uniformly degenerated elliptic equations is studied by [7]. Uniform elliptic operator of the second order of divergent structure is considered in the paper [8].

Let E_n be n dimensional Euclidean space of the points $x = (x_1, ..., x_n)$. Denote by $R > 0$ for $B_R\left(x_R^0\right)$ the ball $\left\{x : \left|x - x^0\right| < R\right\}$, and by $Q_T^R\left(x_R^0\right)$ the cylinder $B_R\left(x^0\right) \cup (0, T)$. Further let for $x^0 \in E_n,\ R > 0$ and $k > 0$ $\varepsilon_{r,k}\left(x^0\right)$ be an ellipsoid $\left\{x : \sum\limits_{i=1}^{n} \frac{\left(x_i - x_i^0\right)^2}{R^{\alpha_i}} < (kR)^2\right\}$. Let D be a bounded domain in E_n with the sufficiently smooth boundary of domain $\partial D,\ 0 \in D$. ε is a such king of ellipsoid that $\overline{D} \subset \varepsilon,\ \mathfrak{B}(\varepsilon)$ is a set of all functions, satisfying in $\overline{\varepsilon}$ the uniform Lipschitz condition and having zero near the $\partial\varepsilon$.

Denote by α and $(\alpha_1, ..., \alpha_n)$ the vector $\langle\alpha\rangle = \alpha_1, ..., \alpha_n$. Condition on α_i is given below.

Denote by $W_{2,\alpha}^1(D)$ the Banach space of the functions $u(x)$ given on D with the finite norm

$$\|u\|_{W_{2,\alpha}^1(D)} = \left(\int\limits_D \left(u^2 + \sum_{i=1}^{n} \lambda_i(x)\, u_i^2\right) dx\right)^{1/2},$$

*Corresponding author: E-mail: tgadjiev@mail.az

where

$$u_i = \frac{\partial u}{\partial x_i},\ i = 1,..,n.\quad \lambda_i(x) = \left(|x|_\lambda\right)^{\alpha_i},\ \ |x|_\alpha = \sum_{i=1}^{n} |x_i|^{\frac{2}{2+\alpha_i}},$$

$$0 \le \alpha_i < \frac{2}{n-1}) \tag{1.1}$$

Further, let $\overset{\circ}{W}{}^{1}_{2,\alpha}(D)$ be a degenerated set of all functions from $C_0^\infty(D)$ by the norm of the space $W^1_{2,\alpha}(D)$. Denote by $\mathcal{M}(D)$ the set of all bounded in D functions.

Let $E \subset D$ be some compact. Denote by $A_E(D)$ the totality of all functions $u(x) \in C^\infty(\overline{D})$, such that $u(x) = 0$ at some neighbourhood of the compact E.

The compact E is called the removable relative to the first boundary value problem for the elliptic operator L in the space $\mathcal{M}(D)$, if all generalized solution of the equation $Lu = 0$ in $D\backslash E$, $u\mid_{\partial D\backslash E}= 0$, $u(x) \in \mathcal{M}(D)$, then $u(x) \equiv 0$ in D. We'll say that the function $u(x) \in \overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)$ is non-negative on the set $H \subset \varepsilon$, in the sense of $\overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)$, if there exists the sequence of the functions $\{u_{(m)}(x)\}, m = 1, 2, ..$, such that $u_m(x) \in \mathfrak{B}(\varepsilon)$, $u_m(x) \ge 0$ for $x \in H$ and $\lim_{m\to\infty} \left\|u_{(m)} - u\right\|_{W^1_{2,\alpha}(\varepsilon)} = 0$.

The function $u(x) \in W^1_{2,\alpha}(D)$ is non-negative on ∂D "in the sense of space" $W^1_{2,\alpha}(D)$, if there exists the sequence of the functions $\{u_m(x)\}, m = 1, 2, ..$, such, that $u_{(m)}(x) \in C^1(D), u_m(x) \ge 0$ for $x \in \partial D$ and $\lim_{m\to\infty} \left\|u_{(m)} - u\right\|_{W^1_{2,\alpha}(\varepsilon)} = 0$. It is easy to determine the inequalities $u(x) \ge const,\ u(x) \ge v(x),\ u(x) \le 0$, and also equality $u(x) = 1$ on the set H in the sense of $\overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)$, if at the same time $u(x) \ge 1$ and $u(x) \le 1$ on H, in the sense of $\overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)$.

Let $\omega(x)$ be a measurable function in D, finite and positive for a.e. $x \in D$. Denote by $L_{p,\omega}(D)$ the Banach space of the functions given on D, with the norm

$$\|u\|_{L_{p,\omega}(D)} = \left(\int_D (\omega(x))^{p/2} |u|^p \, dx\right)^{1/p},\quad 1 < p < \infty.$$

Let $W^1_{p,\alpha}(D)$ be a Banach space of the functions given on $u(x)$, with the finite norm D.

$$\|u\|_{W^1_{p,\alpha}(D)} = \left(\int_D \left(|u|^p + \sum_{i=1}^{n} (\lambda_i(x))^{p/2} |u_i|^p\right) dx\right)^{1/p},\quad 1 < p < \infty$$

Analogously to $\overset{\circ}{W}{}^{1}_{2,\alpha}(D)$, it is introduced the subspace $\overset{\circ}{W}{}^{1}_{p,\alpha}(D)$ for $1 < p < \infty$. The space, conjugated to $\overset{\circ}{W}{}^{1}_{p,\alpha}(D)$ we'll denote by $\overset{*}{W}{}^{1}_{p,\alpha}(D)$.

We'll consider the elliptic operator in the bounded domain $D \subset E_n$

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

In assumption, that $\|a_{ij}(x)\|$ is a real symmetric matrix with measurable in D elements, moreover for all $\xi = (\xi_1, ..., \xi_n) \in E_n$ and $x \in D$ the condition

$$\gamma \sum_{i=1}^{n} \lambda_i(x)\,\xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j \le \gamma^{-1} \sum_{i=1}^{n} \lambda_i(x)\,\xi_i^2 \tag{1.2}$$

is fulfilled. Here $\gamma \in (0, 1]$ is a constant.

The function $u(x) \in W^1_{2,\alpha}(D)$ is called the generalized solution of the equation $Lu = f(x)$ in D, if for any function $\eta(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(D)$ the integral identity

$$\int_D \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i} \eta_{x_j} dx = \int_D f\eta dx \tag{1.3}$$

is fulfilled.

Here $f(x)$ is a given function from $L_2(D)$.

Let $E \subset D$ be some compact. The function $u(x) \in W^1_{2,\alpha}(D \backslash E)$ is called a generalized solution of the equation $Lu = f(x)$ in $D \backslash E$, $u(x) = 0$ on ∂D, if integral identity (1.3) is fulfilled for any function $\eta(x) \in A_E(D)$.

We'll assume that the coefficients of the operator L are continued in $E_n \backslash D$ with conditions (1.1), (1.2). For this, it is enough to assume that $a_{ij}(x) = \delta_{ij}\lambda_i(x)$ for $x \in E_n \backslash D$, $i, j = 1, .., n$, where δ_{ij} is a Croneker symbol.

Let $h(x) \in W^1_{2,\alpha}(D)$, $f^0(x) \in h_2(D)$, $f^i(x) \in L_{2,\lambda^{-1}}(D)$, $i = 1, 2, .., n$, are given functions. Let's consider the first boundary value problem

$$Lu = f^0(x) + \sum_{i=1}^{n} \frac{\partial f^i(x)}{\partial x_i}, \quad x \in D \tag{1.4}$$

$$(u(x) - h(x)) \in \overset{\circ}{W}{}^1_{2,\alpha}(D) \tag{1.5}$$

The function $u(x) \in W^1_{2,\alpha}(D)$ we'll call a generalized solution of problem (1.4)-(1.5) if for any function $\eta(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(D)$ the integral identity

$$\int_D \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i} \eta_{x_j} dx = \int_D \left(-f^0\eta + \sum_{i=1}^{n} f^i \eta_{x_i} \right) dx$$

is fulfilled.

Our aim is to get the necessary and sufficient condition of removability of the compact E.

2 Preliminaries Statements

At first, we introduce some auxiliary statements.

Lemma 2.1. *If relative to the coefficients of the operator L, conditions (1.1), (1.2) be fulfilled, then the first boundary value problem (1.4)-(1.5) has a unique generalized solution $u(x)$ at any $h(x) \in W^1_{2,\alpha}(D)$, $f^0(x) \in h_2(D)$, $f^i(x) \in L_{2,\lambda_i^{-1}}(D)$, $i = 1, 2, .., n,$. At this there exists $P_0(\alpha, n)$ such that, if $p > p_0,\ h(x) \in W^1_{p,\alpha}(D)$, $f^0(x) \in h_p(D)$, $f^i(x) \in L_{2,\lambda_i^{-1}}(D)$, $i = 1, 2, .., n$, $\partial D \in C^1$, then solution $u(x)$ is continuous in $\overline{D}$.*

Lemma 2.2. *Let relative to the coefficients of the operator L conditions (1.1), (1.2) be fulfilled. Then any generalized solution of the equation $Lu = 0$ in D is continuous by Holder at each strictly internal domain ∂.*

Lemma 2.3. *Let relative to the coefficients of the operator L, conditions (1.1), (1.2) be fulfilled and $\overline{\varepsilon_{R,1}} < D$. Then for any positive solution $u(x)$ of the equation $Lu = 0$ in D the Harnack inequality is true*

$$\sup_{\varepsilon_{R,1}(0)} u \le C_1(\gamma, \alpha, n) \inf_{\varepsilon_{R,1}(0)} u \tag{2.1}$$

If at this $y \in \partial\varepsilon_{R,2}(0)$ and $\overline{\varepsilon_{R,1}}(0) \subset D$, then the inequality of form (2.1) is true in ellipsoid $\varepsilon_{R,1}(y)$.

Lemma 2.4. *Let relative to the coefficients of the operator L conditions (1.1), (1.2) be fulfilled, and $u(x)$ be generalized solution of the first boundary-value problem (1.4), (1.5) at $f^i(x) \equiv 0,\ i = 0,..,n.$ Then if $h(x)$ is bounded on ∂D in the sense of $W_{2,\alpha}^1(D)$, then for solution $u(x)$ the following maximum principle is true*

$$\inf_{\partial D} h \leq \inf_{D} u \leq \sup_{\partial D} h,$$

where $\inf_{\partial D} h \left(\sup_{\partial D} h\right)$ is an exact lower (upper) bound of numbers a, for which $h(x) \geq a$ $(h(x) \leq a)$ on ∂D in the sense of $W_{2,\alpha}^1(D)$.

These lemmas are proved as in paper [7].

Let $H \subset \varepsilon$ be some compact, V_H be a set of all functions $\varphi(x) \in \overset{\circ}{W}{}_{2,\alpha}^{1}(\varepsilon)$, such that $\varphi(x) \geq 1$ on H, in the sense of $\overset{\circ}{W}{}_{2,\alpha}^{1}(\varepsilon)$. Let's consider the functional

$$J_\theta(\varphi) = \int_\varepsilon \sum_{i,j=1}^n a_{ij}(x)\,\varphi_i\varphi_j dx, \quad \varphi(x) \in V_H$$

The value $\inf_{\varphi \in V_H} J_\theta(u)$ is called L capacity of the compact H relative to ellipsoid ε and denoted by $cap_L^{(\varepsilon)}(H)$. In case $\varepsilon = E_n$, the corresponding value is called L capacity of the compact H and denoted by $cap_L(H)$.

Lemma 2.5. *There exists the unique function $u(x) \in \overset{\circ}{W}{}_{2,\alpha}^{1}(\varepsilon)$ such that $u(x) \geq 1$ on H in the sense of $\overset{\circ}{W}{}_{2,\alpha}^{1}(\varepsilon)$ and $cap_L^{(\varepsilon)}(H) = J_L(u)$. Moreover, $u(x) = 1$ on H in the sense of $\overset{\circ}{W}{}_{2,\alpha}^{1}(\varepsilon)$.*

Proof. It is easy to see that V_H is convex closed set in $\overset{\circ}{W}{}_{2,\alpha}^{1}(\varepsilon)$. From the fact that $\overset{\circ}{W}{}_{2,\alpha}^{1}(\varepsilon)$ is a Hilbert Space, it follows the existence of unique function $u(x) \in V_H$, on which the functional $J_L(u)$ achieved an exact lower bound. Let $\{u(x)\}^1 = \begin{cases} u(x) & \text{if } u(x) \leq 1 \\ 1 & \text{if } u(x) > 1 \end{cases}$

It is clear, that $\{u(x)\}^1 \in \overset{\circ}{W}{}_{2,\alpha}^{1}(\varepsilon)$. Moreover, $\{u(x)\}^1 \in V_H$. Denote by $A^+ = \{x : x \in \varepsilon,\ u(x) > 1\}$. We have

$$J_L\left\{u(x)^1\right\} = \left(\int_{A^+} + \int_{\varepsilon \setminus A^+}\right) \sum_{i,j=1}^n a_{ij}(x)\,\{u\}_i^1\,\{u\}_j^1\,dx = \int_{\varepsilon \setminus A^+} \sum_{i,j=1}^n a_{ij}(x)\,u_i u_j dx \tag{2.2}$$

On the other side, according to (1.1)

$$\int_{A^+} \sum_{i,j=1}^n a_{ij}(x)\,u_i u_j dx \geq 0 \tag{2.3}$$

From (2.2) and (2.3) we conclude

$$J_L\left\{u(x)^1\right\} \leq J_L(u) = \inf_{\varphi \in V_H} J_L(\varphi)$$

i.e., $J_L\left\{u(x)^1\right\} = J_L(u)$. From uniqueness of extreme function it follows, that $\{u(x)\}^1 = u(x)$, and lemma is proved.

The function $u(x)$,on which the functional $J_L(u)$ achieved its exact lower bound is called L capacity potential of the compact H relative to the ellipsoid ε.

Lemma 2.6. *Let L be a capacity potential of $u(x)$ of the compact H relative to ε. Then $u(x)$ is a generalized solution of the equation $Lu = 0$ in $\varepsilon \backslash H$, tending to 0 on $\partial\varepsilon$ and to 1 on ∂H in the sense of $W^1_{2,\alpha}(\varepsilon)$.*

Proof. It is sufficient to show the truthness of the first part of assertion of lemma. Let $\eta(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ and $\eta(x) \geq 0$ on H in the sense of $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$. Then for any $\varepsilon > 0$ $(u(x) + \varepsilon\eta(x)) \in V_H$. Therefore

$$J_L(u + \varepsilon\eta) \geq J_L(u).$$

Thus

$$J_L(u) + \varepsilon^2 J_L(\eta) + 2\varepsilon \int\limits_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x)\, u_i \eta_j dx \geq J_L(u),$$

i.e.

$$J_L(u) + 2\varepsilon \int\limits_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x)\, u_i \eta_j dx \geq 0.$$

Tending ε to zero, we conclude

$$\int\limits_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x)\, u_i \eta_j dx \geq 0. \tag{2.4}$$

It is easy to see as $\eta(x)$ in (2.4) we can take any function from $C^1(\bar{\varepsilon})$ with compact support in $\varepsilon \backslash H$. Then

$$\int\limits_{\varepsilon\backslash H} \sum_{i,j=1}^{n} a_{ij}(x)\, u_i \eta_j dx \geq 0.$$

Substituting $\eta(x)$ on -$\eta(x)$, we get the equality

$$\int\limits_{\varepsilon\backslash H} \sum_{i,j=1}^{n} a_{ij}(x)\, u_i \eta_j dx = 0$$

Lemma is proved.

Let μ be a charge of bounded variation, given on ε. We'll say, that the function $u(x) \in L_1(\varepsilon)$ is a weak solution of the equation $Lu = -\mu$, equaling to zero on $\partial\varepsilon$, if for any function $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon) \cap C(\bar{\varepsilon})$ the integral identity

$$\int\limits_{\varepsilon} uL\varphi dx = \int\limits_{\varepsilon} \varphi d\mu.$$

is fulfilled.

According to lemma 2.1 (at $h = 0$) there exists the continuous linear operator H from $\overset{*}{W}{}^1_{2,\alpha}(\varepsilon)$ in $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$, such that for any functional $T \in \overset{*}{W}{}^1_{2,\alpha}(\varepsilon)$, the function $u = H(T)$ is an unique generalized solution of the equation $Lu = T$ in $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$.

The operator H is called Green operator.

By lemma 2.1 this operator at $p > p_0$ we transform $\overset{*}{W}{}^1_{2,\alpha}(\varepsilon)$ to $C(\bar{\varepsilon})$. It is easy to see, that the function $u(x)$ is weak solution of the equation $Lu = -\mu$, equaling to zero on $\partial\varepsilon$, iff for any function $\psi(x) \in C(\bar{\varepsilon})$ the integral identity

$$\int\limits_{\varepsilon} u\psi dx = \int\limits_{\varepsilon} H(\psi)\, d\mu. \tag{2.5}$$

is fulfilled.

By analogy with [8] we can show that for each measure μ on ε there exists the unique weak solution of the equation $Lu = -\mu$ equaling to zero on $\partial\varepsilon$.

Let's say, that the charge $\mu \in \overset{*}{W}{}^{1}_{2,\alpha}(\varepsilon)$ if there exists the vector $\overline{f}(x) = \left(f^{\circ}(x), f^{1}(x), ..., f^{n}(x)\right)$ $f^{0}(x) \in h_2(\varepsilon), f^{i}(x) \in L_{2,\lambda_i}(\varepsilon), i = 1, 2, .., n,$ for any function $\varphi(x) \in \overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon) \cap C(\overline{\varepsilon})$ the integral identity

$$\mu(\varphi) = \int\limits_{\varepsilon} \varphi d\mu = \int\limits_{\varepsilon} \left(f^{\circ}\varphi - \sum_{i=1}^{n} f^{i}\varphi_i \right) dx.$$

is true.

So, it is obvious that

$$\left| \int\limits_{\varepsilon} \varphi d\mu \right| \leq C_2\left(\overline{f}\right) \|\varphi\|_{W^{1}_{2,\alpha}(\varepsilon)} \ .$$

Lemma 2.7. *The weak solution $u(x)$ of the equation $Lu = -\mu$, equaling to zero on $\partial\varepsilon$, belongs to $\overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)$, iff $\mu \in \overset{*}{W}{}^{1}_{2,\alpha}(\varepsilon)$*

Proof. At first, we'll show that if the function $\varphi(x) \in \overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)$ satisfies the integral identity

$$\int\limits_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x) u_i \varphi_j dx = -\int\limits_{\varepsilon} \varphi d\mu \tag{2.6}$$

for any function $\varphi(x) \in \overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon) \cap C(\overline{\varepsilon})$, then it is weak solution of the equation $Lu = -\mu$, equaling to zero on $\partial\varepsilon$. Really, assuming $\varphi = H(\psi)$, $\psi(x) \in C(\overline{\varepsilon})$ we obtain

$$\int\limits_{\varepsilon} H(\psi) d\mu = \int\limits_{\varepsilon} \varphi d\mu = -\int\limits_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x) u_i \varphi_j dx$$

$$= \int\limits_{\varepsilon} u \sum_{i,j=1}^{n} \left(a_{ij}(x) \varphi_j\right)_i dx = \int\limits_{\varepsilon} uL\varphi dx = \int\limits_{\varepsilon} u\psi dx,$$

and now it is sufficient to use the identity (2.5). We'll show that $\mu \in \overset{*}{W}{}^{1}_{2,\alpha}(\varepsilon)$. For this, it is sufficient to prove, that if $f^{i}(x) = \sum\limits_{i=1}^{n} a_{ij}(x) u_i(x)$, then $f^{i}(x) \in L_{2,\lambda_i^{-1}}(\varepsilon), i = 1, 2, .., n,$. Assume in condition (2.6) $\xi_1 = ... = \xi_{i-1} = \xi_{i+1} = ... = \xi_n = 0, \xi_i = \dfrac{1}{\sqrt{\lambda_i(x)}}$.

We'll obtain

$$\gamma \leq \frac{a_{ii}(x)}{\lambda_i(x)} \leq \gamma^{-1}; \ \ i = 1, .., n. \tag{2.7}$$

Let $i \neq j$. Assuming $\xi_k = 0$ at $k \neq j$ and $k \neq i$, $\xi_i = \dfrac{1}{\sqrt{\lambda_i(x)}}$, $\xi_j = \dfrac{1}{\sqrt{\lambda_j(x)}}$, we'll obtain

$$2\gamma \leq \frac{a_{ii}(x)}{\lambda_i(x)} + \frac{a_{jj}(x)}{\lambda_j(x)} + \frac{2a_{ij}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}} \leq 2\gamma^{-1}$$

Using (2.7), we conclude

$$\frac{|a_{ij}(x)|}{\sqrt{\lambda_i(x)\lambda_j(x)}} \leq \gamma^{-1} - \gamma; \ \ i, j = 1, ..., n; \ i \neq j \tag{2.8}$$

From (2.7) and (2.8) it follows that

$$\frac{|a_{ij}(x)|}{\sqrt{\lambda_i(x)\lambda_j(x)}} \le \gamma^{-1}; \quad i,j=1,...,n; \tag{2.9}$$

Thus, from (2.9) take out for $j=1,...,n$

$$\int_{\varepsilon} \frac{1}{\lambda_j(x)}\left(f^j\right)^2 dx = \int_{\varepsilon} \frac{1}{\lambda_j(x)}\left(\sum_{i=1}^{n} a_{ij}(x)u_i\right)^2 dx \le \gamma^{-2} n \sum_{i=1}^{n}\int_{\varepsilon} \lambda_i(x)u_i^2 dx < \alpha$$

So, $\mu \in \overset{*}{W}{}^{1}_{2,\alpha}(\varepsilon)$. And vice versa, if $u(x)$ is a weak solution of the equation $Lu=-\mu$, and $u(x)=0$ on $\partial\varepsilon$, then there exists $\mu \in \overset{*}{W}{}^{1}_{2,\alpha}(\varepsilon)$, such that

$$\left(f^{\circ}\varphi - \sum_{i=1}^{n} f^i\varphi_i\right)dx = \int_{\varepsilon}\varphi d\mu = \int_{\varepsilon} uL\varphi dx$$

$$= \int_{\varepsilon} u\sum_{i,j=1}^{n}\left(a_{ij}(x)\varphi_j\right)_i dx = -\int_{\varepsilon}\sum_{i,j=1}^{n} a_{ij}(x)u_i\varphi_j dx$$

for any function $\varphi(x) \in \overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon) \cap C(\overline{\varepsilon}), L\varphi(x) \in C(\overline{\varepsilon})$.

Then, from lemma 2.1 we obtain that $u(x) \in \overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)$. The lemma is proved.

Let now $\delta(x)$ be Dirac measure, concentrated at the point 0, y is an arbitrary fixed point ε.

The weak solution $g(x,y)$ of the equation $Ly=-\delta(x-y)$, such that $g(x,y)=0$ on $\partial\varepsilon$ is called the Green function of the operator L in ε.

In case $\varepsilon = E_n$ the corresponding function is called the fundamental solution of the operator L and denoted by $G(x,y)$.

According to above proved, if $\psi(x)$ is an arbitrary function from $C(\overline{\varepsilon})$, then the generalized solution $\varphi(x) \in \overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)$ of the equation $L\varphi = -\psi$ can be introduced in the following from

$$\varphi(y) = \int_{\varepsilon} g(x,y)\psi(x)dx.$$

We can show, that $g(x,y)$ is non-negative in $\varepsilon\times\varepsilon$, moreover, $g(x,y)=g(y,x)$.

Lemma 2.8. *For any charge, of bounded variation on ε the integral*

$$u(x) = \int_{\varepsilon} g(x,y)d\mu(y)$$

exists, finite a.e. in ε and is weak solution of the equation $Lu=-\mu$, equaling to zero on $\partial\varepsilon$.

Proof. Without losing generality, we'll assume that the charge μ is the measure in ε. Let $\varphi(x) \in C(\overline{\varepsilon})$, $\psi(x) \ge 0$ in ε. Denote by $\varphi(x) \in \overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)$ the generalized solution of the equation $L\varphi = -\psi(x)$. Then $\varphi(x) \in C(\overline{\varepsilon})$ according to lemma 2.1 and $\psi(x) \ge 0$ according to lemma 2.4. So

$$\varphi(y) = \int_{\varepsilon} g(x,y)\psi(x)dx.$$

Then, by Fubini theorem we conclude, that the integral $\int\limits_{\varepsilon} g(x,y)\,d\mu(y)$ there exists for almost all $x \in \varepsilon$, moreover

$$\int\limits_{\varepsilon} H(\psi)\,d\mu(y) = \int\limits_{\varepsilon} \varphi(y)\,d\mu(y) = \iint\limits_{\varepsilon\times\varepsilon} g(x,y)\,\psi(x)\,dxd\mu(y) = \int\limits_{\varepsilon} \psi(x)\,u(x)\,dx. \tag{2.10}$$

Let's note, that the equality (2.10) is fulfilled for weak non-negative and continuous in $\overline{\varepsilon}$ function $\psi(x)$. Now, it is sufficient to remember the identity (2.5) and lemma is proved.

Let's consider now L-capacity of the potential $u(x)$ of the compact H relative to the ellipsoid ε. It was proved above that $u(x)$ satisfies the inequality (2.4) at any non-negative on H the function $\eta(x) \in C_0^\infty(\varepsilon)$. By the Schwartz theorem [9] there exists the measure μ on H such that

$$\int\limits_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x)\,u_i\eta_j\,dx = \int\limits_{\varepsilon} \eta\,d\mu. \tag{2.11}$$

Further, since $u = 1$ on H in the sense of $\overset{\circ}{W}{}^{1}_{2,\alpha}(\varepsilon)$, then the carrier of the measure μ is located on ∂H. The measure μ is called L -capacity distribution of the compact H.

According to lemma 2.8 L-capacity potential $u(x)$ is weak solution of the equation $Lu = -\mu$, equaling to zero on $\partial\varepsilon$ and can be represented in the following form

$$u(x) = \int\limits_{\varepsilon} g(x,z)\,d\mu(z) \tag{2.12}$$

On the other side, there exists the sequence of the functions $\left\{\eta^{(m)}(x)\right\};\quad m = 1,2,...$, such that $\eta^{(m)}(x) \in \mathfrak{B}(\varepsilon)\,, \eta^{(m)}(x) = 1$ for $x \in H$ and
$\lim\limits_{m\to\infty} \left\|\eta^{(m)} - u\right\|_{W^1_{2,\alpha}(\varepsilon)} = 0$. Assuming in equality (2.5) $\eta^{(m)}(x)$ instead of $\eta^{(m)}$, we conclude that the right-hand side is equal to $\mu(H)$ at any natural m, while the left-hand side tends to $cap_L^{(\varepsilon)}(H)$ as $m \to \infty$. Thus,

$$cap_L^{(\varepsilon)}(H) = \mu(H) \tag{2.13}$$

Lemma 2.9. *Let relative to coefficients of the operator L conditions (1.1)-(1.2), $y \in \partial\varepsilon_{R,2}(0)$, $\overline{\varepsilon}_{R,1}(0) \subset D$, $x \in \partial\varepsilon_{R,1}(y)$ be fulfilled. Then for the Green function $g(x,y)$ the following estimations are true*

$$C_3(\gamma,\alpha,n)\left[cap_L^{(\varepsilon)}(\overline{\varepsilon}_{R,1}(y))\right]^{-1} \le g(x,y) \le C_4(\gamma,\alpha,n)\left[cap_L^{(\varepsilon)}(\overline{\varepsilon}_{R,1}(y))\right]^{-1} \tag{2.14}$$

If $\overline{\varepsilon}_{R,1}(0) \subset D$, $x \in \partial\varepsilon_{R,1}(0)$ then

$$C_3\left[cap_L^{(\varepsilon)}(\overline{\varepsilon}_{R,1}(0))\right]^{-1} \le g(x,0) \le C_4\left[cap_L^{(\varepsilon)}(\overline{\varepsilon}_{R,1}(0))\right]^{-1} \tag{2.15}$$

Proof. Without loss of generality, we can assume that the coefficients of the operator L are continuously differentiable in $\overline{\varepsilon}$. The general case is obtained by means of limit passage. Then at $x \neq y$ the function $g(x,y)$ is continuous by x and y, moreover

$$\lim_{x\to y} g(x,y) = \infty) \tag{2.16}$$

Let a be a positive number, which will be chosen later, $K_a = \{x : g(x,y) \ge a\}$, where y is an arbitrary fixed point on $\partial\varepsilon_{R,2}(0)$. From (2.16) it follows that y is an internal point of the compact K_a.

Then L is a capacity potential K_a, represented in form (2.12). So it means, it is equal to zero there. Thus,

$$1 = \int_{\varepsilon} y\left(y, z\right) d\mu_a\left(z\right)$$

where μ is a L-capacity distribution of the compact K_a. Let the carrier of the measure μ_a is located on ∂K_a, where $g\left(y, z\right) = a$. Then using (18), we obtain

$$\mu_a\left(K_a\right) = cap_L^{(\varepsilon)}\left(K_a\right) = \frac{1}{a} \tag{2.17}$$

Let's assume now, $a = \inf\limits_{x \in \partial\varepsilon_{R,1}(y)} g\left(x, y\right)$. According to maximum principle $\overline{\varepsilon}_{R,1}\left(y\right) \subset K_a$. Therefore from (2.17) we conclude

$$cap_L^{(\varepsilon)}\left(\overline{\varepsilon}_{R,1}\left(y\right)\right) \leq cap_L^{(\varepsilon)}\left(K_a\right) = \frac{1}{\inf\limits_{x \in \partial\varepsilon_{R,1}(y)} g\left(x, y\right)} \tag{2.18}$$

If we assume $b = \sup\limits_{x \in \partial\varepsilon_{R,1}(y)} g\left(x, y\right)$, then $\overline{\varepsilon}_{R,1}\left(y\right) \subset K_a$, i.e.,

$$cap_L^{(\varepsilon)}\left(\overline{\varepsilon}_{R,1}\left(y\right)\right) \leq cap_L^{(\varepsilon)}\left(K_b\right) = \frac{1}{\sup\limits_{x \in \partial\varepsilon_{R,1}(y)} g\left(x, y\right)} \tag{2.19}$$

From (2.18) and (2.19) follows that

$$\inf_{x \in \partial\varepsilon_{R,1}(y)} g\left(x, y\right) \leq \left[cap_L^{(\varepsilon)}\left(\overline{\varepsilon}_{R,1}\left(y\right)\right)\right]^{-1} \leq \sup_{x \in \partial\varepsilon_{R,1}(y)} g\left(x, y\right) \tag{2.20}$$

On the other side, according to lemma 2.3

$$\sup_{x \in \partial\varepsilon_{R,1}(y)} g\left(x, y\right) \leq C_5\left(\gamma, \alpha, n\right) \inf_{x \in \partial\varepsilon_{R,1}(y)} g\left(x, y\right) \tag{2.21}$$

Now, the required estimations (2.14) follows from (2.20) and (2.21). Absolutely analogously the truthness of inequality (2.15) is proved.

Corollary 2.10. . *Let the conditions of the lemma, and* $y \in \partial\varepsilon_{R,2}\left(0\right)$ *be fulfilled,* $\overline{\varepsilon}_{R,1}\left(0\right) \subset D$, $x \in \partial\varepsilon_{R,1}\left(0\right)$ *or* $y = 0$, $\overline{\varepsilon}_{R,1}\left(0\right) \subset D$, $x \in \partial\varepsilon_{R,1}\left(0\right)$. *Then for the fundamental solution* $G\left(x, y\right)$ *the estimations*

$$C_3\left[cap_L^{(\varepsilon)}\left(\overline{\varepsilon}_{R,1}\left(0\right)\right)\right]^{-1} \leq G\left(x, y\right) \leq C_4\left[cap_L^{(\varepsilon)}\left(\overline{\varepsilon}_{R,1}\left(0\right)\right)\right]^{-1} \tag{2.22}$$

are true.

3 REMOVABILITY CRITERION OF THE COMPACT IN THE SPACE $M\left(D\right)$

Theorem 3.1. *Let relative to the coefficients of the operator* L*, conditions (1.1)-(1.2) be fulfilled. Then for removability of the compact* $E \subset D$ *relative to the first boundary value problem for the operator* L *in the space* $\mathcal{M}\left(D\right)$ *it is necessary and sufficient, that*

$$cap_L\left(E\right) = 0 \tag{3.1}$$

Proof. Let the ellipsoid ε has the same sense, that above. It is easy to see that if condition (3.1) is fulfilled, then

$$cap_L^{(\varepsilon)}(E) = 0$$

Without loss of generality, we can consider the case, when the coefficients of the operator L is continuously differentiable in $\overline{\varepsilon}$. Let's fix an arbitrary $\varepsilon > 0$ and $x^0 \subset D\backslash E$. By virtue of (3.1) there exists the neighbourhood H of the compact E, such that

$$cap_L^{(\varepsilon)}\left(\overline{H}\right) < \varepsilon \tag{3.2}$$

So, we can assume that ε is such small, that

$$dist\left(x^0, \overline{H}\right) \geq \frac{1}{2} dist\left(x^0, E\right) \tag{3.3}$$

Denote by $V_H(x)$ and μ_H the L-capacity potential of the compact $\overline{H}$ relative to the ellipsoid ε and $\overline{L}$-capacity of the distribution $\overline{H}$, respectively. According to above proved

$$V_H(x) = \int\limits_{\varepsilon} g(x, y)\, d\mu_H(y),$$

moreover the function $V_H(x)$ is a generalized solution of the equation $LV_H = 0$ in $\varepsilon\backslash\overline{H}$, which is equal to 0 on $\partial\varepsilon$ and equal to 1 on ∂H in the sense of $W^1_{2,\alpha}(\varepsilon)$. Let now, $u(x) \in \mathcal{M}(D)$ is an arbitrary solution of the equation $Lu = 0$ in $D\backslash E$, such that $u(x) = 0$ on ∂D. Let $M = \sup\limits_D |u|$. It is easy to see, that the function $V_H(x)$ is non-negative on ∂D, in the sense of $W^1_{2,\alpha}(D)$. Hence, it follows, that the function $u(x) - MV_H(x)$ is non-positive on $\partial\left(D\backslash\overline{H}\right)$ generalized solution of the equation $Lu = 0$ in $D\backslash\overline{H}$. According to lemma 2.4 $u(x) - MV_H(x) \leq 0$ and $D\backslash\overline{H}$ in particular

$$u\left(x^0\right) \leq MV_H\left(x^0\right) \leq M \sup_{y\in\partial H} g\left(x^0, y\right) \mu_H\left(\overline{H}\right) = M \sup_{y\in\partial H} g\left(x^0, y\right) cap_L^{(\varepsilon)}\left(\overline{H}\right) \tag{3.4}$$

By virtue of continuity of the function $g(x, y)$ at $x \neq y$ and inequality (3.3) we obtain

$$\sup_{y\in\partial H} g\left(x^0, y\right) \leq C_6\left(\gamma, \alpha, n, x^0, E\right)$$

Thus, from (3.2) and (3.5) we conclude

$$u\left(x^0\right) \leq MC_6\varepsilon \tag{3.5}$$

Using an arbitrary ε, we get the inequality

$$u\left(x^0\right) \leq 0 \tag{3.6}$$

Making similar considerations with the function $u(x) + MV_H(x)$, we obtain

$$u\left(x^0\right) \geq 0 \tag{3.7}$$

From (3.5)-(3.6) and an arbitrariness of the point x^0 it follows, that $u(x) \equiv 0$ in $D\backslash E$. Thereby, the sufficiency of condition (3.1) is proved. Let's prove its necessity. Let's assume that $cap_L(E) > 0$. Denote by ε' the ellipsoid, such that $\overline{\varepsilon}' \subset \delta$, $E \subset \varepsilon'$. Assume $D = \varepsilon$. Further, let $u_E(x)$ be V_E-L capacity potential of the compact E relative to the ellipsoid ε' and L-capacity distribution E, respectively. Following to [10], we can give the equivalent definition of Vallee-Poussin type of L-capacity of the compact E, relative to the ellipsoid ε'. Let $g(x, y)$ be a Green function of the operator L in ε'. Let's call the measure μ on E, L-admissible, if $\mu \subset E$ and

$$V_\mu^E(x) = \int\limits_{\varepsilon'} g(x, y)\, d\mu(y) \leq 1 \quad for \;\; x \in \sup p\mu \tag{3.8}$$

The value $\sup \mu (E) = cap_L^{(\varepsilon')} (E)$, where an exact upper boundary is taken on all L-admissible measures, is called L-capacity of the compact E, relative to the ellipsoid ε'.

Analogously, the L-capacity $cap_L (E)$ is determined. At this by the standard method we show, that there exists the unique measure, on which an exact upper boundary of the functional $\mu (E)$ is reached, by the set of all L-admissible measures μ. This measure is L-capacity distribution of the compact E.

According to the above proved, the function $u_E (x)$ is generalized solution of the equation $Lu_E = 0$ in $\varepsilon' \setminus E$, equaling to zero on $\partial\varepsilon'$. Besides, from (3.7) and maximum principle it follows that $u_E (x) \in M (\varepsilon')$. On the other side $u_E (x) \not\equiv 0$, as $V_H (E) > 0$. Theorem is proved.

Lemma 3.2. *Let relative to the coefficients of the operator L condition (1.1) be fulfilled. Then, if $y \in \partial\varepsilon_{R,2} (0)$, then $C_7 (\gamma, \alpha, n) R^{n+\frac{\langle\alpha\rangle}{2}-2} \leq cap_L (\overline{\varepsilon}_{R,1} (y)) \leq C_8 (\gamma, \alpha, n) R^{n+\frac{\langle\alpha\rangle}{2}-2}$*

Proof. Let $L_0 = \sum\limits_{i=1}^{n} \dfrac{\partial}{\partial x_i} \left(\lambda_i (x) \dfrac{\partial}{\partial x_i} \right)$. Then, according to (1.1)

$$\gamma cap_{L_0} (\overline{\varepsilon}_{R,1} (y)) \leq cap_L (\overline{\varepsilon}_{R,1} (y)) \leq \gamma^{-1} cap_{L_0} (\overline{\varepsilon}_{R,1} (y)) . \tag{3.9}$$

Let $u (x) \in C_0^{\infty} \left(\varepsilon_{R,\frac{3}{2}} (y) \right)$, $u (x) = 1$ for $\varepsilon_{R,1} (y)$, moreover

$$|u_i (x)| \leq \frac{C_9 (\lambda, n)}{R^{1+\frac{\alpha_i}{2}}}; \;\; i = 1, .., n \tag{3.10}$$

Then

$$cap_{L_0} (\overline{\varepsilon}_{R,1} (y)) \leq \int\limits_{\varepsilon_{R,\frac{3}{2}} (y)} \sum_{i=1}^{n} \lambda_i (x) u_i^2 dx. \tag{3.11}$$

On the other side, as $y \in \partial\varepsilon_{R,2} (0)$, then $\sum\limits_{i=1}^{n} \dfrac{y_i^2}{R^{\alpha_i}} = 4R^2$ and thereby

$$|y_i| \leq 2R^{1+\frac{\alpha_i}{2}}; \;\; i = 1, ..., n.$$

Besides, as $x \in \varepsilon_{R,\frac{3}{2}} (y)$, then

$$|x_i - y_i| \leq \frac{3}{2} R^{1+\frac{\alpha_i}{2}}; \;\; i = 1, ..., n.$$

Thus

$$|x_i| \leq |y_i| + |x_i - y_i| \leq \frac{7}{2} R^{1+\frac{\alpha_i}{2}}; \;\; i = 1, ..., n.$$

Hence, it follows that

$$|x|_{\alpha} \leq R \sum_{i=1}^{n} \left(\frac{z}{2} \right)^{\frac{2}{2+\lambda_i}}$$

Therefore

$$\lambda_i (x) \leq C_{10}^{\alpha_i} R^{\alpha_i} \leq C_{10}^{\alpha^+} R^{\alpha_i}; \; i = 1, ..., n. \tag{3.12}$$

where $\alpha^+ = \max \{\alpha_1, ..., \alpha_n\}$.

Using (3.10) and (3.12) in (3.11) we obtain

$$cap_{L_0} (\overline{\varepsilon}_{R,1} (y)) \leq C_{10} (\alpha, n) R^{-2} mes \left(\varepsilon_{R,\frac{3}{2}} (y) \right) = C_{11} (\alpha, n) R^{n+\frac{\langle\alpha\rangle}{2}-2}$$

and by virtue of (3.9), the upper estimation of (3.8) is proved.

For truthness of lower estimation of (3.8), we note that

$$cap_{L_0}\left(\overline{\varepsilon}_{R,1}(y)\right)\geq cap_{L_0}\left(\overline{\varepsilon}_{R,\frac{1}{2\sqrt{n}}}(y)\right) \tag{3.13}$$

Besides, considering the same as in [8], we conclude

$$cap_{L_0}\left(\overline{\varepsilon}_{R,\frac{1}{2\sqrt{n}}}(y)\right)\geq C_{12}(\alpha,n)\,cap_{L_0}^{(\varepsilon_0)}\left(\overline{\varepsilon}_{R,\frac{1}{2\sqrt{n}}}(y)\right) \tag{3.14}$$

where $\varepsilon_0=\varepsilon_{R,\frac{1}{\sqrt{n}}}(y)$.

Let $W=\left\{u(x):u(x)\,C_0^{\infty}(\varepsilon_0)\,,\ u(x)=1 \text{ for } x\in\varepsilon_{R,\frac{1}{2\sqrt{n}}}(y)\right\}$. Then

$$cap_{L_0}^{(\varepsilon_0)}\left(\overline{\varepsilon}_{R,\frac{1}{2\sqrt{n}}}(y)\right)=\inf_{u\in W}\int_{\varepsilon_0}\sum_{i=1}^{n}\lambda_i(x)\,u_i^2dx. \tag{3.15}$$

On the other side, if $y\in\partial\varepsilon_{R,2}(0)$, then we can find $i_0,\ 1\leq i_0\leq n$, such that $y_{i_0}^2\geq\frac{4R^{2+\alpha_{i_0}}}{n}$, i.e.,

$$|y_{i_0}|\geq\frac{4R^{1+\frac{\alpha_{i_0}}{2}}}{\sqrt{n}}$$

Besides, as $x\in\varepsilon_0$, then

$$|x_{i_0}-y_{i_0}|\leq\frac{R^{1+\frac{\alpha_{i_0}}{2}}}{\sqrt{n}}$$

Therefore

$$|x_{i_0}|\geq|y_{i_0}|-|x_{i_0}-y_{i_0}|\geq\frac{R^{1+\frac{\alpha_{i_0}}{2}}}{\sqrt{n}}$$

Thereby

$$\lambda_i(x)\geq n^{-\frac{1}{2+\alpha_{i_0}}}R;\ \ i=1,...,n. \tag{3.16}$$

where $\alpha^-=\min\{\alpha_1,...,\alpha_n\}$.

Using (3.16) in (3.15) we obtain

$$cap_{L_0}^{(\varepsilon_0)}\left(\overline{\varepsilon}_{R,\frac{1}{2\sqrt{n}}}(y)\right)=C_{13}(\alpha,n)\inf_{u\in W}\int_{\varepsilon_0}\sum_{i=1}^{n}R^{\alpha_i}u_i^2dx. \tag{3.17}$$

Denote by $B_R(z)$ the ball $\{x:|x-z|<R\}$. Let's substitute the variables $v_i=\frac{x_i}{R^{1+\frac{\alpha_i}{2}}};\,i=1,...,n$ in (3.17) and let $\widetilde{y}$ is an image of the point y, where $\widetilde{W}=\left\{\widetilde{u}(v):\widetilde{u}(\tau)\,C_0^{\infty}(B_0)\,,\ \widetilde{u}(\tau)=1 \text{ for } v\in B_{\frac{1}{2\sqrt{n}}}(\widetilde{y})\right\}$. Then from (3.17) we deduce $B_0=B_{\frac{1}{2\sqrt{n}}}(\widetilde{y})$ where by (3.17)

$$cap_{L_0}^{(\varepsilon_0)}\left(\overline{\varepsilon}_{R,\frac{1}{2\sqrt{n}}}(y)\right)\geq C_{13}R^{n+\frac{\langle\alpha\rangle}{2}-2}\inf_{\widetilde{u}\in\widetilde{W}}\int_{B_0}\sum_{i=1}^{n}\left(\frac{\partial\widetilde{u}}{\partial v_i}\right)^2d\tau$$

$$=C_{13}R^{n+\frac{\langle\alpha\rangle}{2}-2}cap^{(B_0)}\left(\overline{B}_{\frac{1}{2\sqrt{n}}}(\widetilde{y})\right),$$

we'll denote by $cap^{(B_0)}\left(\overline{B}_{\frac{1}{2\sqrt{n}}}(\widetilde{y})\right)$ the Wiener capacity of the compact $\overline{B}_{\frac{1}{2\sqrt{n}}}(\widetilde{y})$, relative to the ball B_0. Now, it is sufficient to note that $cap^{(B_0)}\left(\overline{B}_{\frac{1}{2\sqrt{n}}}(\widetilde{y})\right)=C_{14}(n)$ and required estimation follow from (3.13), (3.14) and (3.18). Lemma is proved.

Lemma 3.3. *Let relative to the coefficients of the operator L condition (1.1) be fulfilled. Then*

$$C_{15}(\gamma,\alpha,n)\,R^{n+\frac{\langle\alpha\rangle}{2}-2}\le cap_L\left(\overline{\varepsilon}_{R,1}(y)\right)\le C_{16}(\gamma,\alpha,n)\,R^{n+\frac{\langle\alpha\rangle}{2}-2} \tag{3.18}$$

Upper estimation in (3.19) is proved analogously to the estimation in (3.8). For proof of the lower estimation, it is sufficient to note that $\varepsilon_{R,\frac{1}{4}}(\overline{y})\subset\varepsilon_{R,1}(0)$, i.e. where

$$cap_L\left(\overline{\varepsilon}_{R,\frac{1}{4}}(\overline{y})\right)<cap_L\left(\overline{\varepsilon}_{R,1}(0)\right) \tag{3.19}$$

where $\overline{y}=\left(\frac{1}{2}R^{1+\frac{\alpha}{2}},0,...,0\right)$ and repeat the proof of the previous lemma.

Corollary 3.4. *If conditions (1.1)-(1.2) $y\in\partial\varepsilon_{R,2}(0)$ are fulfilled, then for any $\rho\in(0,R]$ the estimation*

$$cap_L\left(\overline{\varepsilon}_{\rho,1}(\overline{y})\right)\le C_{17}(\gamma,\alpha,n)\,\rho^{n+\frac{\langle\alpha\rangle}{2}-2}\left(1+\sum_{i=1}^{n}\left(\frac{R}{\rho}\right)^{\alpha_i}\right). \tag{3.20}$$

is true.

Then $v(x)\in C_0^\infty\left(\varepsilon_{\rho,\frac{3}{2}}(y)\right),\ \ v(x)=1$ *for* $x\in\varepsilon_{\rho,1}(y)$

$$|v_i(x)|\le\frac{C_{18}(\alpha,n)}{\rho^{1+\frac{\alpha_i}{2}}};\ \ i=1,...,n$$

$$cap_{L_0}\left(\overline{\varepsilon}_{\rho,1}(\overline{y})\right)=\gamma^{-1}C_{18}^2\rho^{-2}\int\limits_{\varepsilon_{\rho,\frac{3}{2}}(y)}\sum_{i=1}^{n}\lambda_i(x)\,\rho^{-\alpha_i}dx. \tag{3.21}$$

On the other side, assuming the same, as well as in the proof of lemma 3.2 we obtain the inequality

$$\lambda_i(x)<C_{19}(\alpha,n)\,(R+\rho)^{\alpha_i}\,,\ \ x\in\varepsilon_{\rho,\frac{3}{2}}(y)\,;\ \ i=1,...,n. \tag{3.22}$$

Now, it is sufficient to take into account that

$$\sum_{i=1}^{n}\left(1+\frac{R}{\rho}\right)^{\alpha_i}\le\sum_{i=1}^{n}\left[1+\left(\frac{R}{\rho}\right)^{\alpha_i}\right]\le n\left(1+\sum_{i=1}^{n}\left(\frac{R}{\rho}\right)^{\alpha_i}\right),$$

and from (3.21)-(3.22) the required estimation (3.20) follows.

Corollary 3.5. *If conditions (1.1)-(1.2) $y\neq 0$, are fulfilled, then at $x\in\varepsilon_{d|y|_d,1}(y)$, $x\neq y$ for the fundamental solution $G(x,y)$ the estimation*

$$G(x,y)\ge C_{20}(\gamma,\alpha,n)\,\frac{\left(|x-y|_\alpha\right)^{2-n-\frac{\langle\alpha\rangle}{2}}}{1+\sum\limits_{i=1}^{n}\left(\frac{|y|_\alpha}{|x-y|_\alpha}\right)^{\alpha_i}} \tag{3.23}$$

is true.

If $y=0$, then estimation (3.23) is true for all $x\neq 0$. Here $d=\frac{1}{n2^{\frac{2}{2+\alpha}}}$.

For proof, at first let's show, that if $y\neq 0$, then $y\notin\varepsilon_{d|y|_d,2}(0)$. Really, as

$$|y|_\alpha=\sum_{i=1}^{n}|y_i|^{\frac{2}{2+\alpha i}} \tag{3.24}$$

then there exists $i_0,\, 1 \le i_0 \le n$, such that

$$|y_0|^{\frac{2}{2+\alpha i_0}} \ge \frac{|y|_\alpha}{n}.$$

Thus

$$\frac{|y_{i_0}^2|}{\left(|y|_\alpha\right)^{\alpha i_0}} \ge \frac{\left(|y|_\alpha\right)^2}{n^{2+\alpha i}}.$$

There by

$$\sum_{i=1}^{n} \frac{y_i^2}{\left(d\,|y|_\alpha\right)^{\alpha_i}} \ge \frac{y_{i_0}^2}{\left(d\,|y|_\alpha\right)^{\alpha_{i_0}}} \ge \frac{\left(d\,|y|_\alpha\right)^2}{(dn)^{2+\alpha i_0}} = \frac{4\left(d\,|y|_\alpha\right)^2}{\left(2^{\frac{2}{2+\alpha i_0}}dn\right)^{2+\alpha i_0}}$$

Now, it is sufficient to note that $2^{\frac{2}{2+\alpha i_0}}dn \le 2^{\frac{2}{2+\alpha}}dn = 1$ and the required assertion is proved. On the other side from (3.24) it follows that for all $i,\, 1 \le i \le n$

$$|y_i|^{\frac{2}{2+\alpha_i}} \le |y|_\alpha \ ,$$

i.e.

$$\sum_{i=1}^{n} \frac{y_i^2}{\left(|y|_\alpha\right)^{\alpha_i}} \le n\left(|y|_\alpha\right)^2.$$

So, we'll show that $\varepsilon_{|y|\alpha,\sqrt{n}}(0)$, if only $y \ne 0$.

Let now, for $y \ne 0,\, x \in \varepsilon_{d|y|_d,1}(y)$ and $x \ne y$. Denote by $|x-y|_\alpha$ the ρ. It is easy to see that there exists $i_1,\, 1 \le i_1 \le n$, such that

$$|x_{i_1} - y_{i_1}|^{\frac{2}{2+\alpha i_1}} \ge \frac{\rho}{n}$$

Hence , it follows that

$$\sum_{i=1}^{n} \frac{(x_i - y_i)^2}{\rho^{\alpha_i}} \ge \frac{(x_{i_1} - y_{i_1})^2}{\rho^{\alpha_1}} \ge \frac{\rho^2}{n^{2+\alpha i_1}} \ge \frac{\rho^2}{n^{2+\alpha}}.$$

Thus $x \notin \varepsilon_{\rho;d_1}(y)$, where $d_1 = \dfrac{1}{n^{1+\frac{\alpha}{2}}}$.Analogously, it is proved that $x \in \varepsilon_{\rho,\sqrt{n}}(y)$. Now, the required estimation (3.23) at $y \ne 0$ follows from (2.22) and corollary 3.4 from lemma 3.2. If $y = 0$, then (3.23), it immediately follows from (2.22) and lemma 2.7.

Let $F(x,y)$ be a positive function, determined in $E_n \times E_n$, continuous at $x \ne y$, moreover $\lim\limits_{x\to y} F(x,y) = \infty$ (condition (A)).

Further, let $E \subset E_n$ be some compact. Let's call the measure μ on E $[F]$ admissible, if $\sup p\mu \subset E$ and $V_\mu^E(x) = \int\limits_E F(x,y)\,d\mu(y) \le 1$, for $x \in \sup p\mu$.

The value $\sup \mu(E) = cap_{[F]}(E)$, where an exact upper boundary is taken by all $[F]$ admissible measures, is called $[F]$-capacity of the compact E.

Theorem 3.6. *Let relative to the coefficients of the operator L conditions (1.1)-(1.2) be fulfilled. Then for removability of the compact $E \subset D$ relative to the first boundary-value problem for the operator L in the space $\mathcal{M}(D)$ it is sufficient that*

$$cap_{[F_1]}(E) = 0 \tag{3.25}$$

where $F_1(x,y) = \left[1 + \sum\limits_{i=1}^{n}\left(\dfrac{|y|\,\alpha}{|x-y|_\alpha}\right)^{\alpha_i}\right]^{-1}\left(|x-y|_\alpha\right)^{2-n-\frac{\langle\alpha\rangle}{2}}.$

Proof. We'll use the following assertion, which is proved in [10]. Let function $F(x, y)$ satisfies condition (A), the compact E has zero $[F]$-capacity, μ zero measure concentrated on E. Then, there exists the point $x^{\circ} \in \sup p\mu$, such that $V_{\mu}^{E}(x^{\circ}) = \infty$. So, the potential of the measure $\sup p\mu$ can't be bounded on any portion B, i.e., for any open set B at $E' \in \sup p\mu \cap B$, the potential $V_{\mu}^{E'}(x)$ is not bound B. In particular, if B is an arbitrary neighbourhood of the point x° that $V_{\mu}^{E'}(x^{\circ}) = \infty$.

Let the condition (3.25) be fulfilled, μ be an arbitrary measure, concentrated on E, $x^{\circ} \in \sup p\mu$ is a point, corresponding to the above-stated assertion at $F = F_1$. Let's assume at first, that $x^{\circ} \neq 0$. Then $|x^{\circ}|_{\alpha} = v > 0$. Further, let B be such small neighborhood of the point x°, that if $E' \in \sup p\mu \cap B$, then

$$\sup_{y \in E'} |y|_{\alpha} \leq (1+\varepsilon)\, r, \quad \inf_{y \in E'} |y|_{\alpha} \geq (1+\varepsilon)\, r,$$

where the number $\varepsilon > 0$ will be chosen later. Let's consider the ellipsoids $\varepsilon_{d|y|_d,1}(y)$ at $y \in E'$. Let's choose ε such small, than $x^0 \in \varepsilon_{d|y|_d,1}(y)$ for all $y \in E'$. Then according to corollary 3.5 from lemma 2.7 we obtain

$$V_{\mu}^{E}\left(x^0\right) = \int_E G\left(x^0, y\right) d\mu(y) \geq \int_{E'} G\left(x^0, y\right) d\mu(y) \geq$$

$$\geq C_{20} \int_E F_1\left(x^0, y\right) d\mu(y) = C_{20} V_{\mu}^{E}\left(x^0\right) = \infty.$$

Hence, it follows that any zero measure μ, concentrated on E can't be L admissible. Thus $cap_L(E) = 0$ and the required assertion follows from theorem 3.1.

Let now $x^{\circ} = 0$. Then, using the equality $G(x, y) = G(y, x)$ and corollary 3.4 from lemma 2.7 we conclude

$$V_{\mu}^{E}(0) = \int_E G(0, y)\, d\mu(y) = \int_E G(y, 0)\, d\mu(y) \geq C_{20} \int_E F_1(y, 0)\, d\mu(y)$$

$$= C_{20} \int_E F_1(0, y)\, d\mu(y) = C_{20} V_{\mu}^{E}(0) = \infty.$$

Theorem is proved.

Remark. Let condition of the real theorem be fulfilled, and the compact $E \subset D$ be removable relative to the first boundary-value problem for the operator L in the space $\mathcal{M}(D)$. Then $mes(E) = 0$.

At first, let's note that the discussion of the proof is the same. As in conclusion of estimation (3.23), we can show that at $x \in \varepsilon_{d|y|_d,1}(y)$, $x \neq y \;\; (y \neq 0)$ and at $x \neq y \; (y = 0)$ the estimations

$$G(x, y) \leq C_{21}(\gamma, \alpha, n) \left(|x - y|_d\right)^{2-n-\frac{\langle\alpha\rangle}{2}} \tag{3.26}$$

is true.

As it was shown in theorem 3.6, if the compact E is a removable, then according to $cap_{[-F_2]}(E) = 0$, where $F_2(x, y) = \left(|x - y|_d\right)^{2-n-\frac{\langle\alpha\rangle}{2}}$.

Hence, it follows that if $mes(E) > 0$, then there exists the point $x^2 \in E$, such that $V^E\left(x^1\right) = \infty$, where

$$V^E(x) = \int_E F_2(x, y)\, dy$$

Moreover, if B' is an arbitrary neighborhood of the point $E' = B' \cap E$, then the potential $V^{E'}(x)$ is not bounded on E'. Let's consider the case $x' \neq 0$. Choose a small neighbourhood B' of the point x^1, that at all $x \in E'$, $y \in E'$ the inequality $|x_i - y_i| \leq 1;\; i = 1, ..., n$ is fulfilled. For $x \in E'$ we have

$$V^{E'}(x) = \int_{E'} \left(\sum_{i=1}^{n} |x_i - y_i|^{\frac{2}{2+\alpha i}}\right)^{2-n-\frac{\langle\alpha\rangle}{2}} dy \leq \int_{E'} \left(\sum_{i=1}^{n} |x_i - y_i|\right)^{2-n-\frac{\langle\alpha\rangle}{2}} dy \leq$$

$$\leq \int\limits_{E'} |x-y|^{2-n-\frac{\langle\alpha\rangle}{2}}\, dy \leq \int\limits_{B''} |z|^{2-n-\frac{\langle\alpha\rangle}{2}}\, dy,$$

where B'' is a ball of the radius $\sqrt{n}$ with the center origin of the coordinate. Now, it is sufficient to note that according to condition (1.2) $\frac{\langle\alpha\rangle}{2} \leq \frac{n}{n-1} \leq \frac{3}{2}$ and the assertion the corollary is proved.

Competing interests

The authors declare that no competing interests exist.

References

[1] Carleson L. Removable singularities of continuous harmonic functions in R^n. Math. Scand. 1963, 12: 15–18.

[2] Moiseev EI. On existence and nonexistence of boundary sets of Neumann problem. Differn. Equat. 1973, 9(5): 901–911.

[3] Landis EM. To the question on uniqueness of solution of the first boundary-value problem for elliptic and parabolic equations of second order. 1978, UMH, 33(3) 151 p.

[4] Gadjiev TS, Mamedova VA. On removable sets of solutions of the second order elliptic and parabolic equations in nondivergent form. Ukr. Math. Journ. 2009, 61(11): 1485–1496.

[5] Harvey R, Polking J. Removable singularities of solutions of linear partial differential equations. Acta Math. 1970, 125: 39-56.

[6] Kilpelainen T, Zhong X. Removable sets for continuous solutions of quasilinear elliptic equations. Proc. Amer. Math. Soc. 2002, 130(6): 1681–1688.

[7] Chanillo S, Wreeden R. Harnack's inequality and mean-value inequalities. Comm. Part Diff. Equat. 1986, 11: 1111–1134.

[8] Littman W, Stampacchia G, Weinberger H. Regular points for elliptic equations with discontinuous coefficients. Ann Scnola Norm. Sup. Pisa 1963, 17: 45–79.

[9] Schwartz L. Theorie des Distributions. "Hermann", 1950 (v.1), 1951 (v.2), Paris.

[10] Landkof NS. Foundations of modern potential theory. "Nauka", Moscow, 1966.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?iid=206&id=6&aid=1168