Fractional Sub-Equation Method and its Applications to the Space–Time Fractional Differential Equations in Mathematical Physics

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> *Research Article*

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Abstract

In the present paper, we construct the analytical solutions of some nonlinear evolution equations involving Jumarie's modified Riemann–Liouville derivative in mathematical physics; namely the space–time fractional modified Benjamin-Bona-Mahony(mBBM) equation and the space–time fractional Zakharov-Kuznetsov-Benjamin-Bona-Mahony(ZKBBM) equation by using a simple method which is called the fractional sub-equation method. As a result, three types of exact analytical solutions are obtained. This method is more powerful and will be used in further works to establish more entirely new solutions for other kinds of nonlinear fractional PDEs arising in mathematical physics.

1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order. In recent years, nonlinear fractional differential equations (FDEs) have been attracted great interest. It is caused by both the development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, engineering, and biology [1–7]. For better understanding the mechanisms of the complicated nonlinear physical phenomena as well as further applying them in practical life, the solution of fractional differential equation [8–15] is much involved. In the past, many analytical and numerical methods have been proposed to obtain solutions of nonlinear FDEs, such as local fractional variational iteration method [16-18], local fractional Adomian decomposition method [19,20], local fractional Fourier series method [21,22], finite difference method [23], finite element method [24], differential transform method [25,26], homotopy perturbation method [27–29] and so on. The fractional differential equations are widely used to describe various complex phenomena in many fields such as the fluid flow,

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signal processing, control theory, systems identification and other areas. Many articles have investigated some aspects of fractional differential equations, such as the existence and uniqueness of solutions to Cauchy type problems, the methods for explicit and numerical solutions, and the stability of solutions [30,31]. Among the investigations for fractional differential equations, research for seeking exact solutions and numerical solutions of fractional differential equations is an important topic, which can also provide valuable reference for other related research.

Recently, Zhang and Zhang [32] introduced a new method called fractional sub-equation method to look for traveling wave solutions of nonlinear FDEs. The method is based on the homogeneous balance principle [33] and Jumarie's modified Riemann-Liouville derivative [34,35]. By using fractional sub-equation method, Zhang et al. successfully obtained traveling wave solutions of nonlinear time fractional biological population model and (4 + 1)-dimensional space-time fractional Fokas equation. More recently, Guo et al. [36] and Lu [37] improved Zhang et al.'s work [32] and obtained exact solutions of some nonlinear FDEs.

In this paper, we will apply the fractional sub-equation method [32] for solving fractional partial differential equations in the sense of modified Riemann–Liouville derivative by Jumarie [34] .To illustrate the validity and advantages of the method, we will apply it to the space-time fractional mBBM equation and the space-time fractional ZKBBM equation.

The rest of this paper is organized as follows. In Section 2, we will describe the Modified Riemann-Liouville derivative and give the main steps of the method here. In Section 3, we give two applications of the proposed method to nonlinear equations. In Section 4, some conclusions are given.

2 Descriptions of Modified Riemann-Liouville Derivative and the Proposed Meth

The Jumarie's modified Riemann–Liouville derivative of order α is defined by the expression [34]

$$
D_x^\alpha f(x) = \begin{cases}
\dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1}(f(\xi)-f(0))d\xi, & \alpha < 0, \\[2ex]
\dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, & 0 < \alpha < 1, \\[2ex]
\left[f^{(\alpha-n)}(x) \right]^{(n)} & n \le \alpha < n+1,\ n \ge 1.
\end{cases}
\tag{1}
$$

Some properties for the proposed modified Riemann–Liouville derivative are listed in [34] as follows:

$$
D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \quad \gamma > 0,
\tag{2}
$$

$$
D_x^\alpha (f(x)g(x)) = g(x)D_x^\alpha f(x) + f(x)D_x^\alpha g(x),
\tag{3}
$$

$$D_x^\alpha f\left[g(x)\right] = f_g'\left[g(x)\right]D_x^\alpha g(x) = D_g^\alpha f\left[g(x)\right](g'(x))^\alpha, \tag{4}$$

The above equations play an important role in fractional calculus in the following sections. We present the main steps of the fractional sub-equation method as follows.

Step 1: Suppose that a nonlinear FDEs, say in two independent variables x and t, is given by

$$P(u, u_x, u_t, D_x^\alpha u, D_t^\alpha u, ...) = 0, \qquad 0 < \alpha \le 1, \tag{5}$$

where $D_x^\alpha u$ and $D_t^\alpha u$ are Jumarie's modified Riemann–Liouville derivatives of u, $u = u(x, t)$ is an unknown function, P is a polynomial in u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 2: By using the traveling wave transformation:

$$u(x, t) = u(\xi), \qquad \xi = kx + ct, \tag{6}$$

where k and c are constants to be determined later, the FDE (5) is reduced to the following nonlinear fractional ordinary differential equation (ODE) for $u = u(\xi)$:

$$P(u, ku', cu', k^\alpha D_\xi^\alpha u, c^\alpha D_\xi^\alpha u, ...) = 0. \tag{7}$$

Step 3: We suppose that Eq. (7) has the following solution:

$$u(\xi) = \sum_{i=0}^n a_i \varphi^i, \tag{8}$$

where $a_i\,(i = 0, 1, 2, ..., n)$ are constants to be determined later, n is a positive integer determined by balancing the highest order derivatives and nonlinear terms in Eq. (5) or Eq. (7) (see Ref.[38] for details), and $\varphi = \varphi(\xi)$ satisfies the following fractional Riccati equation:

$$D_\xi^\alpha \varphi = \sigma + \varphi^2, \tag{9}$$

where σ is a constant. By using the generalized Exp-function method via Mittag-Leffler functions [39], Zhang et al. first obtained the following solutions of fractional Riccati equation (9)

$$\varphi(\xi) = \begin{cases} -\sqrt{-\sigma} \tanh_\alpha(\sqrt{-\sigma}\,\xi), & \sigma < 0, \\[2mm] -\sqrt{-\sigma} \coth_\alpha(\sqrt{-\sigma}\,\xi), & \sigma < 0, \\[2mm] \sqrt{\sigma} \tan_\alpha(\sqrt{\sigma}\,\xi), & \sigma > 0, \\[2mm] -\sqrt{\sigma} \cot_\alpha(\sqrt{\sigma}\,\xi), & \sigma > 0, \\[2mm] -\dfrac{\Gamma(1+\alpha)}{\xi^\alpha + \omega}, & \omega = \text{const.}, \quad \sigma = 0, \end{cases} \tag{10}$$

where the generalized hyperbolic and trigonometric functions are defined as

$$\sinh_\alpha(x) = \frac{E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)}{2}, \quad \cosh_\alpha(x) = \frac{E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)}{2}, \quad \tanh_\alpha(x) = \frac{\sinh_\alpha(x)}{\cosh_\alpha(x)},$$

$$\coth_\alpha(x) = \frac{\cosh_\alpha(x)}{\sinh_\alpha(x)}, \qquad \sin_\alpha(x) = \frac{E_\alpha(ix^\alpha) - E_\alpha(-ix^\alpha)}{2i}, \quad \cos_\alpha(x) = \frac{E_\alpha(ix^\alpha) + E_\alpha(-ix^\alpha)}{2},$$

$$\tan_\alpha(x) = \frac{\sin_\alpha(x)}{\cos_\alpha(x)}, \qquad \cot_\alpha(x) = \frac{\cos_\alpha(x)}{\sin_\alpha(x)},$$

where $E_\alpha(z)$ denotes the Mittag-Leffler function, given as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}.$$

Step 4: Substituting Eq. (8) along with Eq. (9) into Eq. (7) and using the properties of Jumarie's modified Riemann–Liouville derivative (2) –(4), we can get a polynomial in $\varphi(\xi)$. Setting all the coefficients of $\varphi^m \, (m = 0, 1, 2, ...)$ to zero, yields a set of overdetermined nonlinear algebraic equations for $a_i \, (i = 0, 1, 2, ..., n), k$ and c.

Step 5: Assuming that the constants $a_i \, (i = 0, 1, 2, ..., n), k$ and c can be obtained by solving the algebraic equations in Step 4, substituting these constants and the solutions of Eq.(9) into Eq.(8), we can obtain the explicit solutions of Eq.(5) immediately.

<u>Remark:</u> If $\alpha \to 1$, the Riccati equation become $\varphi'(\xi) = \sigma + \varphi^2(\xi)$ used in [40]. So the method in this example can be used to solve integer-order differential equations. In this sense, we would like to conclude that our method includes the existing tanh-function method as special case.

3 Applications of the Method

In this section, we apply the fractional sub-equation method to construct the exact solutions for some nonlinear fractional PDEs, namely the space–time fractional mBBM equation and the

space–time fractional ZKBBM equation which are very important nonlinear fractional PDEs in mathematical physics and have been paid attention by many researchers.

3.1 Example 1 the Space–Time Fractional mBBM Equation

We first consider the space-time fractional mBBM equation [41] in the form:

$$D_t^\alpha u + D_x^\alpha u - \nu u^2 D_x^\alpha u + D_x^{3\alpha} u = 0, \tag{11}$$

where ν is a nonzero positive constant. This equation was first derived to describe an approximation for surface long waves in nonlinear dispersive media. It can also characterize the hydromagnetic waves in cold plasma, acoustic waves in inharmonic crystals and acoustic gravity waves in compressible fluids. By considering the traveling wave transformation (6), Eq. (11) can be reduced to the following nonlinear fractional ODE:

$$c^\alpha D_\xi^\alpha u + k^\alpha D_\xi^\alpha u - \nu k^\alpha u^2 D_\xi^\alpha u + k^{3\alpha} D_\xi^{3\alpha} u = 0. \tag{12}$$

By balancing the highest order derivative terms and nonlinear terms in Eq. (12), we suppose that Eq. (12) has the following formal solution:

$$u(\xi) = a_0 + a_1 \varphi(\xi), \tag{13}$$

where $\varphi(\xi)$ satisfies Eq. (9).

Substituting Eq. (13) along with Eq. (9) into Eq. (12) and then setting the coefficients of $\varphi^i \ (i = 0,1,2,3,4)$ to zero, we can obtain a set of algebraic equations for k, c, a_o, a_1 as follows:

$$\varphi^0 : a_1 c^\alpha \sigma + a_1 k^\alpha \sigma - a_0^2 a_1 k^\alpha \sigma \nu + 2a_1 k^{3\alpha} \sigma^2 = 0,$$
$$\varphi^1 : -2a_0 a_1^2 k^\alpha \sigma \nu = 0,$$
$$\varphi^2 : a_1 c^\alpha + a_1 k^\alpha - a_0^2 a_1 k^\alpha \nu + 8a_1 k^{3\alpha} \sigma - a_1^3 k^\alpha \sigma \nu = 0, \tag{14}$$
$$\varphi^3 : -2a_0 a_1^2 k^\alpha \nu = 0,$$
$$\varphi^4 : 6a_1 k^{3\alpha} - a_1^3 k^\alpha \nu = 0.$$

Solving the algebraic equations (14) by Maple or Mathematica, we have:

$$a_0 = 0, \qquad a_1 = \pm k^\alpha \sqrt{\frac{6}{\nu}}, \qquad \sigma = -\frac{k^\alpha + c^\alpha}{2k^{3\alpha}}. \tag{15}$$

We, therefore, obtain from Eqs. (10), (13) and (15) three types of exact solutions of Eq. (11), namely, two generalized hyperbolic function solutions, two generalized trigonometric function solutions and one rational solution as follows:

$$u = \mp \sqrt{\frac{3(k^{\alpha} + c^{\alpha})}{\nu k^{\alpha}}} \tanh_{\alpha} \left[\sqrt{\frac{k^{\alpha} + c^{\alpha}}{2k^{3\alpha}}} (kx + ct) \right], \quad k^{\alpha} + c^{\alpha} > 0, \quad k^{3\alpha} > 0, \quad (16)$$

$$u = \mp \sqrt{\frac{3(k^{\alpha} + c^{\alpha})}{\nu k^{\alpha}}} \coth_{\alpha} \left[\sqrt{\frac{k^{\alpha} + c^{\alpha}}{2k^{3\alpha}}} (kx + ct) \right], \quad k^{\alpha} + c^{\alpha} > 0, \quad k^{3\alpha} > 0, \quad (17)$$

$$u = \pm \sqrt{\frac{-3(k^{\alpha} + c^{\alpha})}{\nu k^{\alpha}}} \tan_{\alpha} \left[\sqrt{\frac{-(k^{\alpha} + c^{\alpha})}{2k^{3\alpha}}} (kx + ct) \right], \quad k^{\alpha} + c^{\alpha} < 0, \quad k^{3\alpha} > 0, \quad (18)$$

$$u = \mp \sqrt{\frac{-3(k^{\alpha} + c^{\alpha})}{\nu k^{\alpha}}} \cot_{\alpha} \left[\sqrt{\frac{-(k^{\alpha} + c^{\alpha})}{2k^{3\alpha}}} (kx + ct) \right], \quad k^{\alpha} + c^{\alpha} < 0, \quad k^{3\alpha} > 0, \quad (19)$$

$$u = \mp \sqrt{\frac{6}{\nu}} \left[\frac{k^{\alpha} \Gamma(1+\alpha)}{(kx + ct)^{\alpha} + \omega} \right], \quad k^{\alpha} + c^{\alpha} = 0, \quad \omega = \text{const.} \quad (20)$$

As $\alpha \to 1$, these obtained exact solutions give the ones of the standard form equation of the space-time fractional mBBM equation (11).

3.2 Example 2. The Space-Time Fractional ZKBBM Equation

We next consider the following space-time fractional ZKBBM equation [42]

$$D_t^{\alpha} u + D_x^{\alpha} u - 2au D_x^{\alpha} u - b D_t^{\alpha} (D_x^{2\alpha} u) = 0. \quad (21)$$

where a and b are arbitrary constants. It arises as a description of gravity water waves in the long-wave regime. Using the traveling wave transformation(6), Eq.(21) can be reduced to the following nonlinear fractional ODE:

$$c^{\alpha} D_{\xi}^{\alpha} u + k^{\alpha} D_{\xi}^{\alpha} u - 2a k^{\alpha} u D_{\xi}^{\alpha} u - bk^{2\alpha} c^{\alpha} D_{\xi}^{3\alpha} u = 0. \quad (22)$$

According to the method described in Section 2, we suppose that Eq.(22) has the following formal solution:

$$u(\xi) = a_0 + a_1 \varphi(\xi) + a_2 \varphi^2(\xi), \quad (23)$$

where $\varphi(\xi)$ satisfies Eq. (9).

Substituting Eq. (23) along with Eq. (9) into Eq. (22) and collect the coefficients of $\varphi^i (i = 0, 1, 2, 3, 4, 5)$ and set them to be zero, a set of algebraic equations are obtained as follows:

$$\varphi^0 : a_1 c^\alpha \sigma + a_1 k^\alpha \sigma - 2a\, a_0 a_1 k^\alpha \sigma - 2a_1 b\, c^\alpha k^{2\alpha} \sigma^2 = 0,$$

$$\varphi^1 : 2a_2 c^\alpha \sigma - 2a\, a_1^2 k^\alpha \sigma + 2a_2 k^\alpha \sigma - 4a\, a_0 a_2 k^\alpha \sigma - 16 a_2 b\, c^\alpha k^{2\alpha} \sigma^2 = 0,$$

$$\varphi^2 : a_1 c^\alpha + a_1 k^\alpha - 2a\, a_0 a_1 k^\alpha - 6a\, a_1 a_2 k^\alpha \sigma - 8a_1 b\, c^\alpha k^{2\alpha} \sigma = 0,$$

$$\varphi^3 : 2a_2 c^\alpha - 2a\, a_1^2 k^\alpha + 2a_2 k^\alpha - 4a\, a_0 a_2 k^\alpha - 4a\, a_2^2 k^\alpha \sigma - 40 a_2 b\, c^\alpha k^{2\alpha} \sigma = 0,$$

$$\varphi^4 : -6a\, a_1 a_2 k^\alpha - 6a_1 b\, c^\alpha k^{2\alpha} = 0,$$

$$\varphi^5 : -4a\, a_2^2 k^\alpha - 24 a_2 b\, c^\alpha k^{2\alpha} = 0.$$

(24)

Solving the set of algebraic equations (24) by Maple or Mathematica yields

$$a_0 = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \sigma)}{2a}, \qquad a_1 = 0, \qquad a_2 = -\frac{6bc^\alpha k^\alpha}{a}. \tag{25}$$

Finally, from Eqs.(10), (23) and (25) we obtain the following generalized hyperbolic function solutions, generalized trigonometric function solutions and rational solution of Eq. (21)

$$u = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \sigma)}{2a} + \frac{6\sigma bc^\alpha k^\alpha}{a} \tanh_\alpha^2(\sqrt{-\sigma}\,\xi), \quad \sigma < 0, \tag{26}$$

$$u = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \sigma)}{2a} + \frac{6\sigma bc^\alpha k^\alpha}{a} \coth_\alpha^2(\sqrt{-\sigma}\,\xi), \quad \sigma < 0, \tag{27}$$

$$u = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \sigma)}{2a} - \frac{6\sigma bc^\alpha k^\alpha}{a} \tan_\alpha^2(\sqrt{\sigma}\,\xi), \quad \sigma > 0, \tag{28}$$

$$u = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \sigma)}{2a} - \frac{6\sigma bc^\alpha k^\alpha}{a} \cot_\alpha^2(\sqrt{\sigma}\,\xi), \quad \sigma > 0, \tag{29}$$

$$u = \frac{1 + c^\alpha k^{-\alpha}}{2a} - \frac{6bc^\alpha k^\alpha \Gamma^2(1+\alpha)}{a(\xi^\alpha + \omega)^2}, \quad \sigma = 0, \quad \omega = \text{conts}, \tag{30}$$

where $\xi = kx + ct$.

4 Conclusions

In this paper, we have seen that three types of exact analytical solutions including the generalized hyperbolic function solutions, generalized trigonometric function solutions and rational solutions for the space-time fractional mBBM equation and the space-time fractional ZKBBM equation are successfully found out by using the fractional sub-equation method.

From our results obtained in this paper, we conclude that the fractional sub-equation method is powerful, effective and convenient for nonlinear fractional PDEs. Also, the solutions of the proposed nonlinear fractional PDEs in this paper have many potential applications in physics and

engineering. Finally, this method provides a powerful mathematical tool to obtain more general exact analytical solutions of a great many nonlinear fractional PDEs in mathematical physics.

To the best of our knowledge, the solutions obtained in this paper have not been reported in literature.

Competing Interests

Author has declared that no competing interests exist.

References

[1]   Kilbas A, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, Elsevier Science, Amsterdam, the Netherlands. 2006;204:1-523.

[2]   Hilfer R. Applications of Fractional Calculus in Physics. World Scientific Publishing, River Edge, NJ, USA; 2000.

[3]   West BJ, Bologna M, Grigolini P. Physics of Fractal Operators. Springer, New York, NY, USA; 2003.

[4]   Miller KS, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations.John Wiley & Sons, New York, NY, USA; 1993.

[5]   Samko SG, Kilbas AA, Marichev OI. Fractional Integrals and Derivatives. Gordon and Breach Science, Yverdon, Switzerland; 1993.

[6]   Podlubny I. Fractional Differential Equations, Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA. 1999;198.

[7]   Oldham KB, Spanier J. The Fractional Calculus, Academic Press, New York, NY, USA; 1974.

[8]   Kiryakova V. Generalized Fractional Calculus and Applications. Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow, UK. 1994;301.

[9]   Podlubny I. Fractional Differential Equations. Mathematics in Science and Engineering, Academic Press, New York, NY, USA. 1999;198.

[10]   Sabatier J, Agrawal OP, Machado JAT. Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, Springer, New York, NY, USA; 2007.

[11]   Mainardi F. Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models, Imperial College Press, London, UK; 2010.

[12] Baleanu D, Diethelm K, Scalas E, Trujillo JJ. Fractional Calculus: Models and Numerical Methods. Series on Complexity, Nonlinearity and Chaos, World Scientific Publishing, Boston, Mass, USA. 2012;3.

[13] Yang XJ, Local Fractional Functional Analysis and Its Applications. Asian Academic Publisher, Hong Kong ; 2011.

[14] Yang XJ. Advanced Local Fractional Calculus and Its Applications. World Science Publisher, New York, NY, USA; 2012.

[15] Ali AHA. The modified extended tanh-function method for solving coupled MKdV and coupled Hirota-Satsuma coupled KdV equations. Phys. Lett. A. 2007;363(5-6):420-425.

[16] Yang XJ, Baleanu D. Fractal heat conduction problem solved by local fractional variation iteration method. Thermal Science; 2012.

[17] Yang XJ. Local fractional Kernel transform in fractal space and its applications. Advances in Comput. Math. Appl. 2012;1(2):86-93.

[18] Yang XJ, Zhang FR. Local fractional variational iteration method and its algorithms. Advances in Comput. Math. Appl.2012;1(3):139-145.

[19] Yang XJ, Zhang YD. A new Adomian decomposition procedure scheme for solving local fractional Volterra integral equation. Advances in Information. Tech. Manag. 2012;1(4):158-161.

[20] Yang XJ, Baleanu D, Zhong WP. Approximation solutions for diffusion equation on Cantor time-space. Proceeding of the Romanian Academy. Series A. 2013:(in press).

[21] Hu MS, Agarwal RP, Yang XJ. Local fractional Fourier series with application to wave equation in fractal vibrating string. Abstract. Appl. Anal. 2012(2012).

[22] Liao MK., Yang XJ, Yan Q. A new viewpoint to Fourier analysis in fractal space. Advances in Appl. Math and Approximation Theory: Contributions from AMAT 2012, Anastassiou GA, Duman O, Ed. Springer, New York, USA. 2012;Chap.26:399-411.

[23] Li C, Chen A, Ye J. Numerical approaches to fractional calculus and fractional ordinary differential equation. J. Comput .Phys. 2011;230(9):3352-3368.

[24] Gao GH, Sun ZZ, Zhang YN. A finite difference scheme for fractional sub-diffusion equations on an unbounded domain using artificial boundary conditions. J. Comput .Phys. 2012;231(7):2865-2879.

[25] Deng W. Finite element method for the space and time fractional Fokker-Planck equation. SIAM J. Numer. Anal. 2008/09;47(1):204-226.

[26] Momani S, Odibat Z, Erturk VS. Generalized differential transform method for solving a space- and time-fractional diffusion-wave equation. Phys. Lett. A. 2007;370(5-6):379-387.

[27] Wu GC, Lee EWM. Fractional variational iteration method and its application. Phys. Lett. A. 2010;374(25):2506-2509.

[28] He J-H, Homotopy perturbation technique, Comput Methods. Appl. Mech. Eng. 1999;178(3-4):257-262.

[29] Fan E. Soliton solutions for a generalized Hirota-Satsuma coupled KdV equation and a coupled MKdV equation. Phys. Lett. A. 2001;282(1-2)18-22.

[30] Saadatmandi A, Dehghan M. A new operational matrix for solving fractional- order differential equations. Comput. Math. Appl. 2010;59(3):1326-1336.

[31] Zhou Y, Jiao F, Li J. Existence and uniqueness for p-type fractional neutral differential equations. Nonlinear Anal. 2009;71(7-8):2724-2733.

[32] Zhang S, Zhang HQ. Fractional sub-equation method and its applications to nonlinear fractional PDEs. Phys. Lett. A. 2011;375(7):1069-1073.

[33] Wang ML. Solitary wave solutions for variant Boussinesq equations. Phys. Lett. A. 1995;199(3-4):169-172.

[34] Jumarie G. Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results. Comput. Math. Appl. 2006;51(9-10)1367-1376.

[35] Jumarie G. Fractional partial differential equations and modified Riemann-Liouville derivative new methods for solution. J. Appl. Math. Comput. 2007;24(1-2):31-48.

[36] Guo S, Mei L, Li Y, Sun Y. The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics. Phys. Lett. A. 2012;376(4):407-411.

[37] Lu B. Bäcklund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations. Phys. Lett. A. 2012;376(28-29):2045-2048.

[38] Zhou YB, Wang ML, Wang YM. Periodic wave solutions to a coupled KdV equations with variable coefficients. Phys. Lett. A. 2003;308(1):31-36.

[39] Zhang S, Zong QA, Liu D, Gao Q. A generalized Exp-function method for fractional Riccati differential equations. Commun. Fract. Calc. 2010;1(1):48-51.

[40] Lü ZS, Zhang HQ. On a new modified extended tanh-function method. Commun. Theor. Phys. (Beijing, China). 2003;39(4):405-408.

[41] Zayed EME, Al-Joudi S. Applications of an extended (G'/G)-expansion method to find exact solutions of nonlinear PDEs in mathematical physics. Math. Probl. Eng. 2010;2010:1-19.

[42] Wazwaz AM. The extended tanh method for new compact and noncompact solutions for the KP-BBM and the ZK-BBM equations. Chaos Solitons Fract. 2008;38(5):1505-1516.

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