Asian Journal of Probability and Statistics

5(3): 1-9, 2019; Article no.AJPAS.51978

ISSN: 2582-0230

The Treatment of Second Order Ordinary Differential Equations Using Equidistant One-step Block Hybrid

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Authors' contributions

This work was carried out in collaboration among all authors. Author YS formulate the method. Author JS studied the basic properties of the method and author MM test the efficiency of the method on some stiff IVPs. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/AJPAS/2019/v5i330136
Editor(s):
(1) Dr. Manuel Alberto M. Ferreira, Professor, Department of Mathematics, ISTA-School of Technology and Architecture, Lisbon University, Portugal.
Reviewers:
(1) Fidele Ngwane, University of South Carolina Salkehatchie, USA.
(2) Mfon Okon Udo, Akwa Ibom State University, Nigeria.
(3) Hamid El Ouardi, University of Hassan II Casablanca, Morocco.
Complete Peer review History: https://sdiarticle4.com/review-history/51978

Original Research Article

Received: 31 July 2019
Accepted: 03 October 2019
Published: 15 October 2019

Abstract

A general one-step hybrid block method with equidistant of order 6 has been successfully developed for the direct solution of second order IVPs in this article. Numerical analysis shows that the developed method is consistent and zero-stable which implies its convergence. The analysis of the new method is examined on two highly and mildly stiff second-order initial value problems to illustrate the efficiency of the method. It is obvious that our method performs better than the existing method within which we compare our result with. Hence, the approach is an adequate one for solving special second order IVPs.

Keywords: One-step; equidistant; IVPs; numerical analysis; highly & mildly second order.

1 Introduction

In this article, we consider an approximate solution of general second order initial value problem (IVP) of the form

$$y'' = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0 \tag{1}$$

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where f is continuously differentiable on the given interval. Seeking an approximate solution for equation (1) is of great importance due to the wide application of this kind of differential equations in science, engineering and other real life problem [1].

Several numerical methods were developed on the hands of many scholars to approximate the solution of problem (1) such as [1-10].

In order to achieve better accuracy and reduce execution time, hybrid block methods were first introduced according to [11,12] and later by [13], while hybrid methods were initially introduced to overcome zero stability barrier occurred in block methods mentioned by Dahlquists [14]. Furthermore, this method have the ability to change step-size besides utilizing data off-step points which contribute to the accuracy of the methods. In hybrid block methods, step and off-step points are combined to form a single block for solving ODEs (see [15-17]). Meanwhile, some scholars such as, [1,17] proposed an implicit one-step hybrid block method for the direct solution of second order ordinary differential equation. Their work motivated us to propose the treatment of second order ordinary differential equation using one-step block with equal equidistant for solving second order IVPs directly of the form (1) using interpolation and collocation.

2 Derivation of the Method

This section shows the development of the second derivative method with single step. The discrete scheme for the block method is constructed from the linear multistep method of the form

$$y_{n+k} = h\sum_{j=0}^{k-1}\alpha_j y_{n+j} + h^2\sum_{j=0}^{k}\beta_j f_{n+j} \tag{2}$$

Using Taylor series expansion to expand individual terms in (2) and upon substitution of the expansions back in (2), the matrix form can be written as below where the coefficients of $y^{(m)}x_n$ are equated

$$\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\left(\frac{2h}{5}\right)^1}{1!} & \frac{\left(\frac{3h}{5}\right)^1}{1!} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\left(\frac{2h}{5}\right)^2}{2!} & \frac{\left(\frac{3h}{5}\right)^2}{2!} & 1 & 1 & 1 & 1 & 1 & 1 \\
\frac{\left(\frac{2h}{5}\right)^3}{3!} & \frac{\left(\frac{3h}{5}\right)^3}{3!} & 0 & \frac{\left(\frac{h}{5}\right)^1}{1!} & \frac{\left(\frac{2h}{5}\right)^1}{1!} & \frac{\left(\frac{3h}{5}\right)^1}{1!} & \frac{\left(\frac{4h}{5}\right)^1}{1!} & \frac{(h)^1}{1!} \\
\frac{\left(\frac{2h}{5}\right)^4}{4!} & \frac{\left(\frac{3h}{5}\right)^4}{4!} & 0 & \frac{\left(\frac{h}{5}\right)^2}{2!} & \frac{\left(\frac{2h}{5}\right)^2}{2!} & \frac{\left(\frac{3h}{5}\right)^2}{2!} & \frac{\left(\frac{4h}{5}\right)^2}{2!} & \frac{(h)^2}{2!} \\
\frac{\left(\frac{2h}{5}\right)^5}{5!} & \frac{\left(\frac{3h}{5}\right)^5}{5!} & 0 & \frac{\left(\frac{h}{5}\right)^3}{3!} & \frac{\left(\frac{2h}{5}\right)^3}{3!} & \frac{\left(\frac{3h}{5}\right)^3}{3!} & \frac{\left(\frac{4h}{5}\right)^3}{3!} & \frac{(h)^3}{3!} \\
\frac{\left(\frac{2h}{5}\right)^6}{6!} & \frac{\left(\frac{3h}{5}\right)^6}{6!} & 0 & \frac{\left(\frac{h}{5}\right)^4}{4!} & \frac{\left(\frac{2h}{5}\right)^4}{4!} & \frac{\left(\frac{3h}{5}\right)^4}{4!} & \frac{\left(\frac{4h}{5}\right)^4}{4!} & \frac{(h)^4}{4!} \\
\frac{\left(\frac{2h}{5}\right)^7}{7!} & \frac{\left(\frac{3h}{5}\right)^7}{7!} & 0 & \frac{\left(\frac{h}{5}\right)^5}{5!} & \frac{\left(\frac{2h}{5}\right)^5}{5!} & \frac{\left(\frac{3h}{5}\right)^5}{5!} & \frac{\left(\frac{4h}{5}\right)^5}{5!} & \frac{(h)^5}{5!}
\end{pmatrix}
\begin{pmatrix}
\alpha_{\frac{2}{5}} \\ \alpha_{\frac{3}{5}} \\ \beta_0 \\ \beta_{\frac{1}{5}} \\ \beta_{\frac{2}{5}} \\ \beta_{\frac{3}{5}} \\ \beta_{\frac{4}{5}} \\ \beta_1
\end{pmatrix}
=
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & \frac{\left(\frac{h}{5}\right)^1}{1!} & \frac{\left(\frac{4h}{5}\right)^1}{1!} & \frac{(h)^1}{1!} \\
0 & \frac{\left(\frac{h}{5}\right)^2}{2!} & \frac{\left(\frac{4h}{5}\right)^2}{2!} & \frac{(h)^2}{2!} \\
0 & \frac{\left(\frac{h}{5}\right)^3}{3!} & \frac{\left(\frac{4h}{5}\right)^3}{3!} & \frac{(h)^3}{3!} \\
0 & \frac{\left(\frac{h}{5}\right)^4}{4!} & \frac{\left(\frac{4h}{5}\right)^4}{4!} & \frac{(h)^4}{4!} \\
0 & \frac{\left(\frac{h}{5}\right)^5}{5!} & \frac{\left(\frac{4h}{5}\right)^5}{5!} & \frac{(h)^5}{5!} \\
0 & \frac{\left(\frac{h}{5}\right)^6}{6!} & \frac{\left(\frac{4h}{5}\right)^6}{6!} & \frac{(h)^6}{6!} \\
0 & \frac{\left(\frac{h}{5}\right)^7}{7!} & \frac{\left(\frac{4h}{5}\right)^7}{7!} & \frac{(h)^7}{7!}
\end{pmatrix}$$

The values of $\left(\alpha_{\frac{2}{5}}, \alpha_{\frac{3}{5}}, \beta_0, \beta_{\frac{1}{5}}, \beta_{\frac{2}{5}}, \beta_{\frac{3}{5}}, \beta_{\frac{4}{5}}, \beta_1\right)^T$ are obtained using matrix inverse method as given below

$$\begin{pmatrix} 3,-2, & \frac{h^2}{375}, & \frac{257h^2}{6000}, & \frac{49h^2}{750}, & \frac{31h^2}{3000}, & -\frac{h^2}{750}, & \frac{h^2}{6000} \\ 2,-1, & -\frac{h^2}{6000}, & \frac{h^2}{250}, & \frac{97h^2}{3000}, & \frac{h^2}{250}, & -\frac{h^2}{6000}, & 0 \\ -1,\ 2, & 0, & -\frac{h^2}{6000}, & \frac{h^2}{250}, & \frac{97h^2}{3000}, & \frac{h^2}{250}, & -\frac{h^2}{6000} \\ -2,\ 3, & \frac{h^2}{6000}, & -\frac{h^2}{750}, & \frac{31h^2}{3000}, & \frac{49h^2}{750}, & -\frac{257h^2}{6000}, & \frac{h^2}{375} \end{pmatrix}^T \tag{3}$$

Substituting (3) in (2) gives the discrete scheme

$$\begin{pmatrix} y_n \\ y_{n+\frac{1}{5}} \\ y_{n+\frac{4}{5}} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 3,-2, & \frac{h^2}{375}, & \frac{257h^2}{6000}, & \frac{49h^2}{750}, & \frac{31h^2}{3000}, & -\frac{h^2}{750}, & \frac{h^2}{6000} \\ 2,-1, & -\frac{h^2}{6000}, & \frac{h^2}{250}, & \frac{97h^2}{3000}, & \frac{h^2}{250}, & -\frac{h^2}{6000}, & 0 \\ -1,\ 2, & 0, & -\frac{h^2}{6000}, & \frac{h^2}{250}, & \frac{97h^2}{3000}, & \frac{h^2}{250}, & -\frac{h^2}{6000} \\ -2,\ 3, & \frac{h^2}{6000}, & -\frac{h^2}{750}, & \frac{31h^2}{3000}, & \frac{49h^2}{750}, & -\frac{257h^2}{6000}, & \frac{h^2}{375} \end{pmatrix} \begin{pmatrix} y_{n+\frac{2}{5}} \\ y_{n+\frac{3}{5}} \\ f_n \\ f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{pmatrix} \tag{4}$$

Following the same approach, which yield the following derivatives of the discrete scheme

$$\begin{pmatrix} y'_n \\ y'_{n+\frac{1}{5}} \\ y'_{n+\frac{2}{5}} \\ y'_{n+\frac{3}{5}} \\ y'_{n+\frac{4}{5}} \\ y'_{n+1} \end{pmatrix} = \begin{pmatrix} -\frac{5}{h}, \frac{5}{h}, & -\frac{397h}{6300}, & \frac{14101h}{50400}, & -\frac{659h}{6300}, & -\frac{1739h}{25200}, & \frac{61h}{3150}, & -\frac{149h}{50400} \\ -\frac{5}{h}, \frac{5}{h}, & \frac{149h}{50400}, & -\frac{257h}{3150}, & -\frac{5429h}{25200}, & -\frac{13h}{6300}, & -\frac{47h}{10080}, & \frac{h}{1260} \\ -\frac{5}{h}, \frac{5}{h}, & -\frac{h}{1260}, & \frac{347h}{50400}, & -\frac{463h}{6300}, & -\frac{191h}{5040}, & \frac{19h}{3150}, & -\frac{37h}{50400} \\ -\frac{5}{h}, \frac{5}{h}, & \frac{37h}{50400}, & -\frac{19h}{3150}, & \frac{191h}{5040}, & \frac{463h}{6300}, & -\frac{347h}{50400}, & \frac{h}{1260} \\ -\frac{5}{h}, \frac{5}{h}, & -\frac{h}{1260}, & \frac{47h}{10080}, & \frac{13h}{6300}, & \frac{5429h}{25200}, & \frac{257h}{3150}, & -\frac{149h}{50400} \\ -\frac{5}{h}, \frac{5}{h}, & \frac{149h}{50400}, & -\frac{61h}{3150}, & \frac{1739h}{25200}, & \frac{659h}{6300}, & -\frac{14101h}{50400}, & \frac{397h}{1260} \end{pmatrix} \begin{pmatrix} y_{n+\frac{2}{5}} \\ y_{n+\frac{3}{5}} \\ f_n \\ f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{pmatrix} \tag{5}$$

Adopting matrix inverse method $y_{n+\frac{1}{5}}, y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, y_{n+\frac{4}{5}}, y_{n+1}, y'_{n+\frac{1}{5}}, y'_{n+\frac{2}{5}}, y'_{n+\frac{3}{5}}, y'_{n+\frac{4}{5}}, y'_{n+1}$ are determined and expressed as shown below

$$\begin{pmatrix} y_{n+\frac{1}{5}} \\ y_{n+\frac{2}{5}} \\ y_{n+\frac{3}{5}} \\ y_{n+\frac{4}{5}} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} y_n + \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{3}{5} \\ \frac{4}{5} \\ 1 \end{pmatrix} hy'_n + h^2 \begin{pmatrix} \frac{1231}{126000}, & \frac{863}{50400}, & -\frac{761}{63000}, & \frac{941}{126000}, & -\frac{341}{126000}, & \frac{107}{252000} \\ \frac{71}{3150}, & \frac{544}{7875}, & -\frac{37}{1575}, & \frac{136}{7875}, & -\frac{101}{15750}, & \frac{8}{7875} \\ \frac{123}{3500}, & \frac{3501}{28000}, & -\frac{9}{3500}, & \frac{87}{2800}, & -\frac{9}{875}, & \frac{9}{5600} \\ \frac{376}{7875}, & \frac{1424}{7875}, & \frac{176}{7875}, & \frac{608}{7875}, & -\frac{16}{1575}, & \frac{16}{7875} \\ \frac{61}{1008}, & \frac{475}{2016}, & \frac{25}{504}, & \frac{125}{1008}, & \frac{25}{1008}, & \frac{11}{2016} \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} y'_{n+\frac{1}{5}} \\ y'_{n+\frac{2}{5}} \\ y'_{n+\frac{3}{5}} \\ y'_{n+\frac{4}{5}} \\ y'_{n+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} y'_n + h \begin{pmatrix} \frac{19}{288}, & \frac{1427}{7200}, & -\frac{133}{1200}, & \frac{241}{3600}, & -\frac{173}{7200}, & \frac{3}{800} \\ \frac{14}{225}, & \frac{43}{150}, & \frac{7}{225}, & \frac{7}{225}, & -\frac{1}{75}, & \frac{1}{450} \\ \frac{51}{800}, & \frac{219}{800}, & \frac{57}{400}, & \frac{57}{400}, & -\frac{21}{800}, & \frac{3}{800} \\ \frac{14}{225}, & \frac{64}{225}, & \frac{8}{75}, & \frac{64}{225}, & \frac{14}{225}, & 0 \\ \frac{19}{288}, & \frac{25}{96}, & \frac{25}{144}, & \frac{25}{144}, & \frac{25}{96}, & \frac{19}{288} \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{pmatrix} \tag{7}$$

3 Analysis of our Block Method

3.1 Order of the block

Let the linear operator $L\{y(x):h\}$ on (6) be

$$y_{n+k} = h \sum_{j=\frac{2}{5},\frac{3}{5}} \alpha_j y_{n+j} + h^2 \sum_{j=0,\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1} \beta_j f_{n+j} \tag{8}$$

Expanding y_{n+j} *and* f_{n+j} in Taylor series and comparing the coefficients of h gives

$$L\{y(x):h\} = C_0 y(x) + C_1 y'(x) + \cdots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + C_{p+2} h^{p+2} y^{p+2}(x) + \cdots$$

Definition 3.1: The linear operator L and the associate block method are said to be of order p if $C_0 = C_1 = \cdots = C_p = C_{p+1} = 0, \quad C_{p+2} \neq 0.$ C_{p+2} is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+2} h^{p+2} y^{p+2}(x)$, [11].

Comparing the coefficients of h, the order of the method (6) is $[6, 6, 6, 6, 6]^T$ with error constants

$$C_8 = \left[-\frac{199}{9450000000}, \quad -\frac{19}{369140625}, \quad -\frac{141}{369140625}, \quad -\frac{8}{73828125}, \quad -\frac{11}{75600000} \right]^T$$

3.2 Consistency

Definition 3.2: A block method is said to be consistent if its order is greater than one. From the above analysis, it is obvious that our method is consistent [14].

3.3 Zero stability

A block is said to be zero stable if as $h \to 0$ if the root of the first characteristic polynomial $\rho(r) = 0$ satisfied $\left| \left[\sum A^0 R^{k-1} \right] \right| \leq 1$, and those roots with $|R| = 1$ must be simple. For our method,

$$\rho(r) = \left| r \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \left[\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] \right| = 0$$

$r^4(r-1) = 0 \Leftrightarrow r = 0, 0, 0, 0, 1$ showing that our method is zero stable [11].

3.4 Convergent

Definition 3.3: The necessary and sufficient conditions for a linear multistep method to be convergent are that it must be consistent and zero stable. Hence our method is convergent [14].

3.5 Region of absolute stability

The region of Absolute stability of the derived block method was plotted using idea of [14].

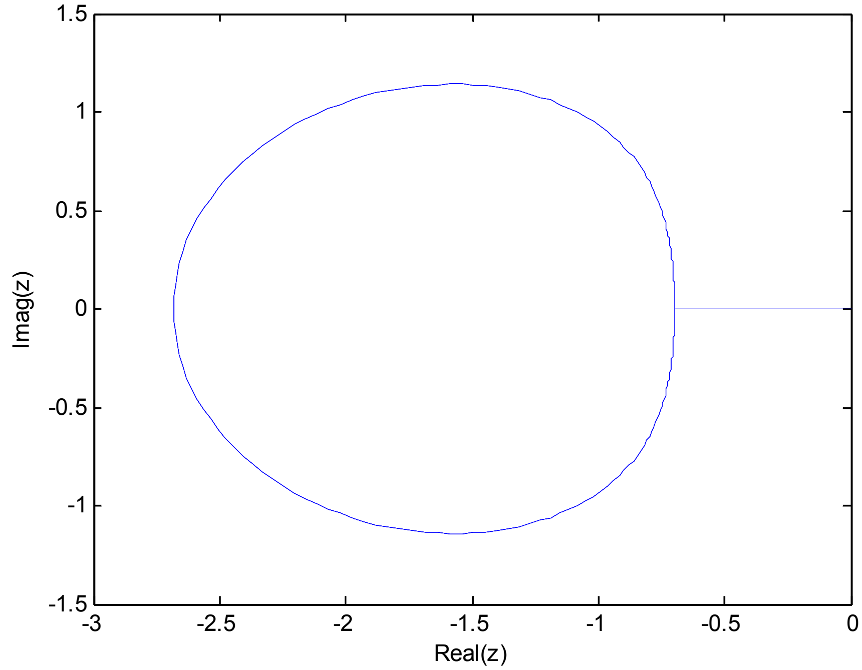

Fig. 1. The absolute stability region of our method

4 Numerical Experiment

The following second order initial value problems of ODEs are considered in order to examine the accuracy of the new developed method.

Problem 1. Solve $f(x, y, y') = -1001y' - 1000y$ such that $y(0) = 1,\ y'(0) = -1$

With exact solution, $y(x) = e^{-x}$ with $h = \frac{1}{10}$

Source: [17 and 6].

Table 1. Comparing the proposed method with [17,6] on problem 4.1

x-values	Exact solution	Computed solution	Error in our method	Error in [17]	Error in [6]
0.100	0.90483741803595957316	0.90483741803595957316	1.998600(-16)	1.054712E(-14)	2.050000E(-11)
0.200	0.81873075307798185867	0.81873075307798185867	1.274500(-16)	1.776357E(-14)	4.390000E(-11)
0.300	0.74081822068171786607	0.74081822068171786607	2.177100(-16)	2.342571E(-14)	6.550000E(-11)
0.400	0.67032004603563930074	0.67032004603563930074	1.927400(-16)	2.797762E(-14)	8.380000E(-11)
0.500	0.60653065971263342360	0.60653065971263342360	2.328600(-16)	3.130829E(-14)	9.860000E(-11)
0.600	0.54881163609402643263	0.54881163609402643263	2.235300(-16)	3.397282E(-14)	1.100000E(-11)
0.700	0.49658530379140951470	0.49658530379140951470	2.396400(-16)	3.563816E(-14)	1.190000E(-11)
0.800	0.44932896411722159143	0.44932896411722159143	2.340900(-16)	3.674838E(-14)	1.240000E(-11)
0.900	0.40656965974059911188	0.40656965974059911188	2.380800(-16)	3.730349E(-14)	1.280000E(-11)
1.000	0.36787944117144232160	0.36787944117144232160	2.323300(-16)	3.741452E(-14)	1.300000E(-11)

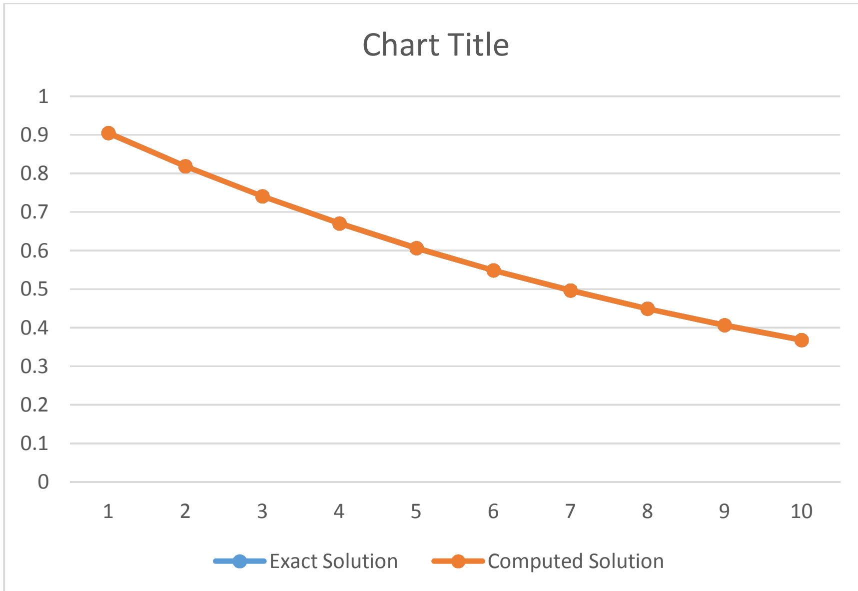

Fig. 2. Graphical solution of Problem 1

Problem 2. Solve $f(x, y, y') = y'$ such that $y(0) = 0,\ y'(0) = -1$

With exact solution, $y(x) = 1 - e^{-x}$ with $h = \dfrac{1}{100}$

Source: [18].

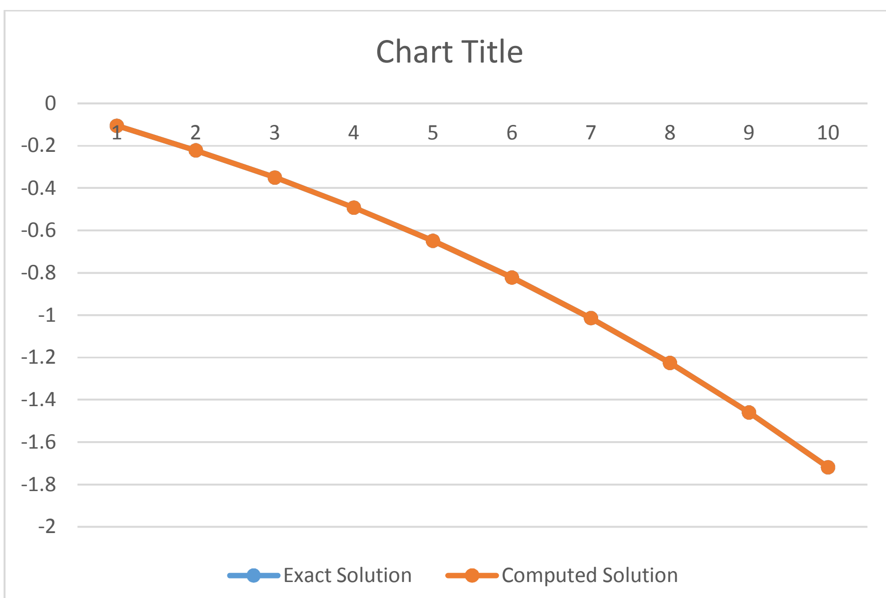

Fig. 3. Graphical solution of Problem 2

Table 2. Comparing the proposed method with [18] on problem 4.2

x-values	Exact Solution	Computed Solution	Error in our method	Error in [18]
0.01	-0.10517091807564762480	-0.10517091807564922592	1.601120E(-15)	2.858824E(-15)
0.02	-0.22140275816016983390	-0.22140275816017659074	6.756840E(-15)	1.439682E(-12)
0.03	-0.34985880757600310400	-0.34985880757601930108	1.619708E(-14)	5.591383E(-11)
0.04	-0.49182469764127031780	-0.49182469764130108397	3.076617E(-14)	4.796602E(-09)
0.05	-0.64872127070012814680	-0.64872127070017958518	5.143838E(-14)	1.003781E(-08)
0.06	-0.82211880039050897490	-0.82211880039058831113	7.933623E(-14)	1.590163E(-08)
0.07	-1.01375270747047652160	-1.01375270747059227280	1.157512E(-13)	2.870014E(-08)
0.08	-1.22554092849246760460	-1.22554092849262977050	1.621659E(-13)	4.284730E(-08)
0.09	-1.45960311115694966380	-1.45960311115716994500	2.202812E(-13)	5.857869E(-08)
0.10	-1.71828182845904523540	-1.71828182845933728000	2.920446E(-13)	8.449297E(-08)

5 Conclusion

A general one-step hybrid block method with equal off-mesh point of order 6 has been successfully developed for the direct solution of general second order IVP.

Numerical analysis shows that the developed method is consistent and zero-stable which implies its convergence. The analysis of the new method is examined on two highly stiff second-order initial value problems to illustrate the efficiency of the method, and we further shown the graph of exact solution and computed result. It is obvious that our method performs better than the existing method of [6,17,18]. Hence, the approach is an adequate one for solving special second order initial value problems.

Competing Interests

Authors have declared that no competing interests exist.

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