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# Solution of a Linear Third order Multi-Point Boundary Value Problem using RKM

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## Abstract

**Aims:** In this paper an approximate method for the solution of third-order differential equation with two and three point boundary condition is developed using iterative reproducing kernel method.

**Methods:** The third order boundary value problem is converted into integro-differential equation of second order two point boundary value problem.

The reproducing kernel method which takes the form of a convergent series with easily computable components is used for the solution of second order two point boundary value problem.

**Results:** Six numerical examples are given to demonstrate the efficiency of the present method. The results obtained are better than the existing methods developed in [19,20,21,22,23].

**Conclusions:** In this paper, the solution of linear and nonlinear third order (two and three point) boundary value problem is determined. For the solution of third order three point boundary value problem reproducing kernel method is proposed and obtained a good accuracy in absolute errors. As the reproducing kernel method cannot solve the third order three-point boundary value problems directly, so the third order boundary value problem is converted to second order two point boundary value problem after absorbing the nonlocal condition at $\alpha$. The method developed is compared with those developed by Li et al. [19], El-Salam et al. [20], Khan and Aziz [21], Li and Wu [22] and Wu and Li [23]. As observed in Example 3, 4, 5 and 6 that the method obtained in this paper is better than [19]-[23]. Results obtained using the scheme presented here show that the numerical scheme is very effective and convenient for third-order linear as well as nonlinear boundary value problem.

Keywords: Approximate solution, gram-Schmidt orthogonal process, reproducing kernel space, integro-differential equation, nonlinear.

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# 1 Introduction

Boundary value problems manifest themselves in many branches of science. For example engineering, technology, control, optimization theory, draining and coating flows and various dynamic systems. In the solution of real-world problems ordinary differential equations (ODEs) are supposed to be basic tools. The problems containing multi-points boundary value developed in different fields of science such that applied mathematics, physics and engineering [1-4]. These multi-points boundary value problems basically originate in mathematical modeling. Two points boundary value problems are widely investigated in literature [5-7]. Graef et al. [8] developed the sufficient conditions for the existence and non-existence of positive solutions for three-point boundary value problems. Boundary value problems are encountered in many engineering fields including optimal control, beam deflections, heat flow, draining and coating flows, and various dynamic systems. In this paper third-order boundary value problems are concerned, such problems arise in the study of draining and coating flows.

Accurate and fast numerical solution of boundary value problems is necessary in many important scientific and engineering applications, e.g. boundary layer theory, the study of stellar interiors, control and optimization theory, and flow networks in biology. Siddiqi and Ghazala [9] proposed a non-polynomial spline method for the numerical solution of the fifth-order linear special case boundary value problems. Siddiqi and Ghazala [10] proposed a sextic spline method for the numerical solution of the fifth order linear special case boundary value problems. Siddiqi et al. [11] developed quintic spline method for the numerical solutions of linear special case sixth-order boundary value problems. The theory of reproducing Kernel has wide applications in numerical analysis, differential equations, probability, statistics and many more [12,13]. A reproducing kernel Hilbert space is a useful framework for constructing approximate solutions of BVPs [14-18].

Consider the following third order boundary value problem:

$$\begin{cases} a_0(x)u^{(3)}(x) + a_1(x)u^{(2)}(x) + a_2(x)u^{(1)}(x) + a_3(x)u(x) = f(x, u(x)), \quad 0 \le x \le 1, \qquad (1.1) \\ u(0) = \alpha_1, u^{(1)}(0) = \alpha_1, u^{(1)}(1) = \alpha u^{(1)}(\eta) + \gamma \end{cases}$$

where $\eta \in (0,1), a_i(x), i = 0,1,2,3$ are continuous on [0, 1].

Remark: The solutions satisfy the conditions mentioned in [8].

## 2 The Conversion of Equation (1.1)

Eq. (1.1) can be converted into an equivalent second order differential equation, which can easily be solved using RKM.

Integrating on both sides of Eq. (1.1) from 1 to $x$, gives

$$\begin{cases} a_0(x)u^{(2)}(x) - a_0(1)u^{(2)}(1) + (a_1(x) + a_0^{(1)}(x))u^{(1)}(x) + (a_0^{(1)}(1) - a_1(1))u^{(1)}(1) + (a_0^{(2)}(x) \\ - a_1^{(1)}(1) + a_2(x))u(x) - (a_0^{(2)}(1) - a_1^{(1)}(1) + a_2(1))u(1) - \int_1^x (a_0^{(3)}(s) - a_1^{(2)}(s) + a_2^{(1)}(s) \\ - a_3(s)u(s))ds = g(x, u(x)), \qquad\qquad\qquad\qquad 0 \le x \le 1, \qquad (2.1) \\ u(0) = \alpha_1, u^{(1)}(0) = \alpha_1, \end{cases}$$

where

Substituting $u^{(1)}(1) = \alpha u^{(1)}(\eta) + \gamma$ into Eq. (2.1), yields

$$
\begin{cases}
a_0(x)u^{(2)}(x) - a_0(1)u^{(2)}(1) + (a_1(x) + a_0^{(1)}(x))u^{(1)}(x) + \alpha(a_0^{(1)}(1) - a_1(1))u^{(1)}(\eta) + (a_0^{(2)}(x) \\
- a_1^{(1)}(1) + a_2(x))u(x) - (a_0^{(2)}(1) - a_1^{(1)}(1) + a_2(1))u(1) - \int_1^x (a_0^{(3)}(s) - a_1^{(2)}(s) + a_2^{(1)}(s) \\
- a_3(s)u(s))ds = h(x, u(x)), \qquad\qquad\qquad 0 \le x \le 1, \qquad (2.2) \\
u(0) = \alpha_1, u^{(1)}(0) = \alpha_1,
\end{cases}
$$

where $h(x, u(x)) = g(x, u(x)) - \gamma(a_0^{(1)}(1) - a_1(1))$.

Assuming *x=1* in Eq. (2.2) and $b(1) - a_0^1(1) \ne 0$, leads to $u^{(1)}(1) = \alpha u^{(1)}(\eta) + \gamma$.

Hence Eq. (2.2) and Eq. (1.1) are equivalent. It may be noted that the solution of Eq. (2.2) gives the solution of Eq. (1.1).

# 3 Reproducing Kernel Hilbert Space Method

(i) The reproducing kernel space $W_2^3[0,1]$ is defined by

$W_2^3[0,1] = \{u/u, u^{(1)}, u^{(2)}$ are absolutely continuous real valued functions in [0,1], $u^{(3)} \in L^2[0,1]\}$.

The inner product and norm in $W_2^3[0,1]$ are given by

$$
\langle u(x), v(x) \rangle = \sum_{i=0}^{2} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(3)} v^{(3)} \, dx, \qquad (3.1)
$$

$$
\|u\| = \sqrt{\langle u, u \rangle}, \quad u, v \in W_2^3[0,1]. \qquad (3.2)
$$

**Theorem 3.1**

The space $W_2^3[0,1]$ is a reproducing kernel Hilbert space. That is, $\forall u \in W_2^3[0,1]$ and each fixed $x, y \in [0,1]$ there exists $R_x(y) \in W_2^3[0,1]$ s.t $\langle u(y), R_x(y) \rangle = u(x)$ and $R_x(y)$ is called the reproducing kernel function of space $W_2^3[0,1]$.

The reproducing kernel function $R_x(y)$ is given by

$$
R_x(y) = \begin{cases}
R_{1x}(y) = \sum_{i=0}^{5} a_i y^i, & y \le x, \\
R_{2x}(y) = \sum_{i=0}^{5} b_i y^i, & y > x,
\end{cases} \qquad (3.3)
$$

where

$$R_{1x}(y) = \frac{1}{20}y^2(y^3 - 5xy + 10x^2(y+3)),$$

$$R_{2x}(y) = R_{1y}(x).$$

Let $Lu(x) = h(x, u(x))$, then clearly $L : W_2^3[0,1] \to W_2^1[0,1]$ is a linear bounded operator.

Using the adjoint operator $L^*$ of $L$ and choose a countable dense subset $T = \{x_1, x_2, ..., x_n, ...\} \subset [0,1]$

and let $\varphi_i(y) = Q_{x_i}(y)$, $i \in N$ then $\psi_i(x) = L^*\varphi_i(x)$, where $\psi_i \in W_2^3[0,1]$.

**Lemma 3.1** If $\{x_i\}_{i=1}^{\infty}$ is dense in [0, 1], then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system of $W_2^3[0,1]$. .

Proof: For each fixed $u \in W_2^3[0,1]$ let $\langle u(x), \psi_i(x) \rangle = 0, i = 1,2,...$ which implies

$$\left\langle u(x), (L^*\varphi_i)(x) \right\rangle = \left\langle (Lu)(x), Q_{x_i}(x) \right\rangle = (Lu)(x_i) = 0.$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense in [0, 1], so $Lu(x) = 0$, which implies $u \equiv 0$ from the existence of $L^{-1}$.

**Lemma 3.2** $\psi_i(x) = L_y R_x(y)|_{y=x_i}$ .

Proof: As

$$\psi_i(x) = L^*\varphi_i(x)$$
$$= \left\langle L^*\varphi_i(y), R_x(y) \right\rangle.$$

Since $L^*$ is conjugate operator, so

$$\psi_i(x) = \left\langle \varphi_i(y), L_y R_x(y) \right\rangle = \left\langle R_{x_i}(y), L_y R_x(y) \right\rangle = L_{x_i} R_x(x_i) = L_y R_x(y)|_{y=x_i} \qquad (3.4)$$

To orthonormalize the sequence $\{\psi_i(x)\}_{i=1}^{\infty}$ in the reproducing kernel space $W_2^3[0,1]$, Gram-Schmidt orthonormalization process can be used, as

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \qquad i = 1,2,3,...$$

$$(3.5)$$

**Theorem 3.2** $\{x_i\}_{i=1}^{\infty}$ is dense on [0, 1], and the solution of the Eq. (2.2) is unique, then the solution is given as

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k, u(x_k))\bar{\psi}_i(x).$$

**Proof:** Using lemma 3.1, it is clear that $\left\{\overline{\psi}_i\right\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W_2^3[0,1]$. Notice that $\left\langle u(x),\phi_i(x)\right\rangle = u(x_i)$, for each $u \in W_2^1[0,1]$. The exact solution can be determined as follows

$$u(x) = \sum_{i=1}^{\infty}\left\langle u(x), \overline{\psi}_i(x)\right\rangle \overline{\psi}_i(x)$$

$$= \sum_{i=1}^{\infty}\left\langle u(x), \sum_{k=1}^{i}\beta_{ik}\psi_k(x)\right\rangle \overline{\psi}_i(x)$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle Lu(x),\varphi_k(x)\right\rangle \overline{\psi}_i(x).$$

If $u$ is the exact solution of Eq. (2.2) and $Lu(x) = h(x,u(x))$, then

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}h(x_k,u(x_k))\overline{\psi}_i(x)$$

The approximate solution obtained by the *n*-term intercept of the exact solution *u*, given by

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}h(x_k,u(x_k))\overline{\psi}_i(x) \qquad (3.6)$$

If the problem (2.2) is nonlinear, then approximate solution of the problem (2.2) can be obtained using the following iteration formula:

$$\begin{cases} Any \quad fixed \quad u_0 \in W_2^3[0,1], \\ u_n(x) = \sum_{i=1}^{n}A_i\overline{\psi}_i(x), \end{cases} \qquad (3.7)$$

where

$$\left.\begin{array}{l} A_1 = \beta_{11}h(x_1,u_0(x_1)) \\ A_2 = \sum_{k=1}^{2}\beta_{2k}h(x_k,u_{k-1}(x_k)) \\ \vdots \\ A_n = \sum_{k=1}^{n}\beta_{nk}h(x_k,u_{k-1}(x_k)) \end{array}\right\} \qquad (3.8)$$

**Theorem 3.3** If

(i) $\left\|u\right\|$ is bounded and

$$\{x_i\}_{i=0}^{\infty}$$

(ii)        is dense in [0,1]

(iii)$h(x, u(x)) \in W_2^1[0,1]$ *and* $u \in W_2^3[0,1]$ then $u_n$ in Eq. (3.7) converges to the exact solution $u$ of

the problem (2.2), where $A_i$ are given by Eq. (3.8) and

$$u(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x),$$

**Proof** (i) From Eq. (3.7), it can be written as

$$u_{n+1}(x) = u_n(x) + A_{n+1} \overline{\psi}_{n+1}(x).$$

Using the orthonormality condition of $\{\overline{\psi}(x)\}_{i=0}^{\infty}$ yields

$$\left\| u_{n+1} \right\|^2 = \left\| u_n \right\|^2 + \left\| A_{n+1} \right\|^2 = \sum_{i=1}^{n+1} A_i.$$

$$\sum_{i=1}^{\infty} A_i < \infty$$

From boundedness of $\left\| u_n \right\|$ gives $\sum_{i=1}^{\infty} A_i < \infty$ *i.e* $\{A_i\} \in L^2, i = 1,2,...$

For $m > n, (u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp (u_{n+1} - u_n)$ leads to

$$\left\| u_m - u_n \right\|^2 = \left\| u_m - u_{m-1} + u_{m-1} - u_{m-2} + ... + u_{n+1} - u_n \right\|^2$$

$$= \left\| u_m - u_{m-1} \right\|^2 + \left\| u_{m-1} - u_{m-2} \right\|^2 + ... + \left\| u_{n+1} - u_n \right\|^2$$

$$= \sum_{i=n+1}^{m} (A_i)^2 \to 0, \ (n, m \to 0)$$

Considering the completeness of $W_2^3[0,1]$ , there exists $u \in W_2^3[0,1]$, such that

$$u_n \to u, n \to \infty$$

(ii) Using (i) of Theorem 3.3, $u_n$ converge uniformly to $u$. On taking limits in Eq. (3.7), it follows

that

$$u(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x)$$

It may be noted that

$$Lu(x_j) = \sum_{i=1}^{\infty} A_i \left\langle L\overline{\psi}_i(x), \varphi_j(x) \right\rangle$$

$$= \sum_{i=1}^{\infty} A_i \left\langle \overline{\psi}_i(x), L^* \varphi_j(x) \right\rangle$$

$$= \sum_{i=1}^{\infty} A_i \left\langle \overline{\psi}_i(x), \psi_j(x) \right\rangle$$

Moreover,

$$\sum_{i=1}^{n}\beta_{nj}Lu(x_j) = \sum_{i=1}^{\infty}A_i\left\langle \overline{\psi}_i(x), \sum_{j=1}^{n}\beta_{ij}\psi_j(x)\right\rangle$$

$$= \sum_{i=1}^{\infty}A_i\left\langle \overline{\psi}_i(x), \overline{\psi}_j(x)\right\rangle = A_n.$$

If $n = 1$, then

$$Lu(x_1) = h(x_1, u_0(x_1))$$

If $n = 2$, then

$$\beta_{21}Lu(x_1) + \beta_{22}Lu(x_2) = \beta_{21}h(x_1, u_o(x_1)) + \beta_{22}h(x_2, u_1(x_2))$$

It is clear that

$$Lu(x_2) = h\left(x_2, u_1(x_2)\right)$$

Furthermore, it is easy to see by induction that

$$Lu(x_j) = h\left(x_j, u_{j-1}(x_j)\right) \tag{3.9}$$

Since, $\{x\}_{i=1}^{\infty}$ is dense on interval [0, 1] and for any $y \in [0, 1]$, there exists subsequence $\{x_{nj}\}$ such that

$$x_{nj} \to y, \ y \to \infty$$

so by the convergence of $u_n$

$$Lu(x) = h(x, u(x)), \tag{3.10}$$

which shows that $u$ is the solution of the problem (2.2) and

$$u_n(x) = \sum_{i=1}^{n}A_i\overline{\psi}_i(x), \tag{3.11}$$

where $A_i$ are given by Eq. (3.8).

## 4 Numerical Examples

To illustrate the applicability and effectiveness of our method, six numerical examples are considered in this section.

**Example 1** Consider the following third order three-point boundary value problem

$$\left. \begin{array}{l} u^{(3)}(x) + u^{(2)}(x) + u^{(1)}(x) = 2\cos x - 2\sin x - x\sin x, \qquad 0 \le x \le 1 \\ u(0) = 0,\ u^{(1)}(0) = 0,\ u^{(1)}(1) = u^{(1)}\left(\dfrac{1}{2}\right) + \dfrac{1}{41}. \end{array} \right\} \qquad (4.1)$$

The exact solution of the problem (4.1) is $u(x) = x\sin x$. The numerical results for n = 30, 50, 100, are summarized in Table 1 and also illustrates by Figs 1 – 3.

**Table 1. The absolute errors when $(n = 30,\ 50,\ 100)$**

| $x$ | Absolute errors (n=30) | Absolute errors (n=50) | Absolute errors (n=100) |
|-----|------------------------|------------------------|-------------------------|
| 0.0 | 0.000000 | 0.000000 | 0.000000 |
| 0.1 | 1.64E-06 | 5.73E-07 | 1.40E-07 |
| 0.2 | 6.59E-06 | 2.31E-06 | 5.65E-07 |
| 0.3 | 1.48E-05 | 5.21E-06 | 1.27E-06 |
| 0.4 | 2.63E-05 | 9.24E-06 | 2.26E-06 |
| 0.5 | 4.09E-05 | 1.43E-05 | 3.51E-06 |
| 0.6 | 5.83E-05 | 2.04E-05 | 5.01E-06 |
| 0.7 | 7.81E-05 | 2.73E-05 | 6.71E-06 |
| 0.8 | 1.01E-04 | 3.51E-05 | 8.58E-06 |
| 0.9 | 1.23E-04 | 4.32E-05 | 1.05E-05 |
| 1.0 | 1.47E-04 | 5.17E-05 | 1.26E-05 |

**Example 2** Consider the following third order three-point boundary value problem

$$\left. \begin{array}{l} u^{(3)}(x) + xu^{(2)}(x) = -6x^2 + 3x - 6 \qquad\qquad 0 \le x \le 1 \\ u(0) = 0,\ u^{(1)}(0) = 0,\ u^{(1)}(1) = u^{(1)}\left(\dfrac{1}{2}\right) + \dfrac{3}{4}. \end{array} \right\} \qquad (4.2)$$

The exact solution of the problem (4.2) is $u(x) = \dfrac{3}{2}x^2 - x^3$. The numerical results are given by Table 2 and illustrates by Figs. 4-6.

**Example 3** Consider the following third order linear boundary value problem [19, 20]

$$\left. \begin{array}{l} u^{(3)}(x) - xu(x) = (x^3 - 2x^2 - 5x - 3)e^x, \qquad 0 \le x \le 1 \\ u(0) = 0, u^{(1)}(0) = 1, u^{(1)}(1) = -e. \end{array} \right\} \qquad (4.3)$$

The exact solution of the problem (4.3) is $u(x) = x(1-x)e^x$. The numerical results are given in Table 3.

**Example 4** Consider the following third order nonlinear boundary value problem [21]

$$u^{(3)}(x) + 2e^{-3u(x)} = 4(1+x)^{-3}, \qquad 0 \le x \le 1$$
$$u(0) = 0, \ u^{(1)}(0) = 1, \ u^{(1)}(1) = \frac{1}{2}. \qquad\qquad\qquad (4.4)$$

The exact solution of the problem (4.4) is $u(x) = \ln(1+x)$. The comparisons of the errors in absolute values between the method developed in this paper and that of [21] are shown in Table 4.
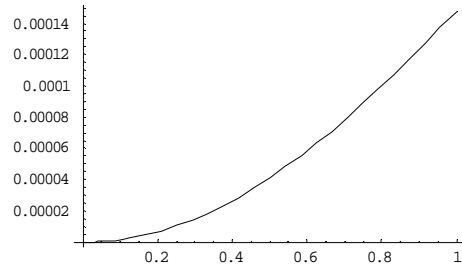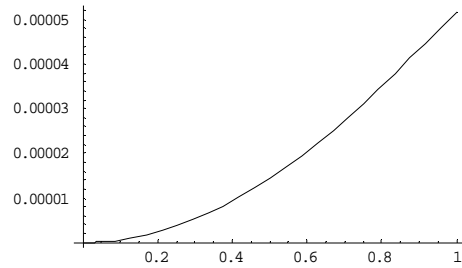


**Fig. 1. |u-u$_{30}$|**



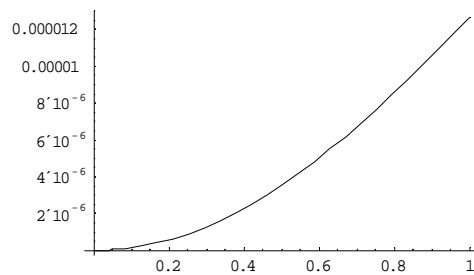**Fig. 2. |u-u$_{50}$|**



**Fig. 3. |u-u$_{100}$ |**
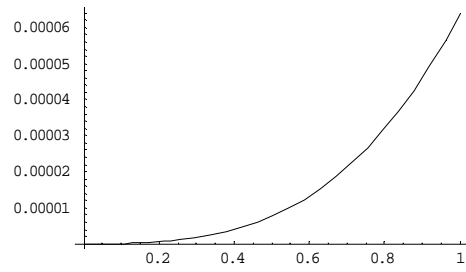
**Fig. 4. |u-u$_{30}$|**
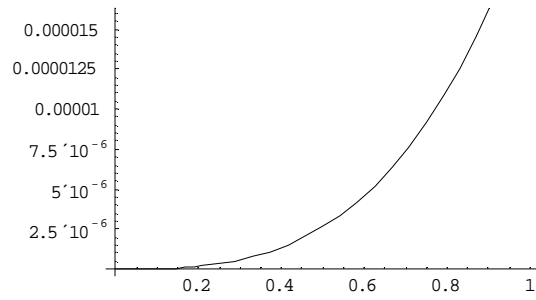


**Fig. 5. |u-u$_{50}$|**
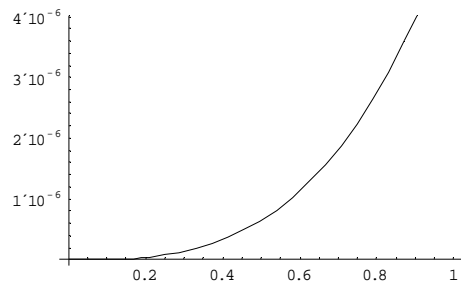


**Fig 6: |u-u$_{100}$|**

**Table 2. The numerical results when ($n = 30, 50, 100$)**

| $x$ | Absolute errors ($n=30$) | Absolute errors ($n=50$) | Absolute errors ($n=100$) |
|---|---|---|---|
| 0.0 | 0.000000 | 0.000000 | 0.000000 |
| 0.1 | 6.29E-08 | 2.13E-08 | 5.17E-09 |
| 0.2 | 4.86E-07 | 1.69E-07 | 4.13E-08 |
| 0.3 | 1.63E-06 | 5.71E-07 | 1.39E-07 |
| 0.4 | 3.88E-06 | 1.35E-06 | 3.32E-07 |
| 0.5 | 7.62E-06 | 2.66E-06 | 6.53E-07 |
| 0.6 | 1.32E-05 | 4.63E-06 | 1.13E-06 |
| 0.7 | 2.12E-05 | 7.42E-06 | 1.81E-06 |
| 0.8 | 3.19E-05 | 1.11E-05 | 2.73E-06 |
| 0.9 | 4.60E-05 | 1.61E-05 | 3.94E-06 |
| 1.0 | 6.39E-05 | 2.24E-05 | 5.48E-06 |

**Table 3. Comparison of numerical results**

| $x$ | Absolute errors (n=20) present method | Absolute errors (n=20) [19] | Absolute errors (n=80) present method | Absolute errors (n=80) [19] |
|---|---|---|---|---|
| 0.0 | 0 | 0 | 0 | 0.000000 |
| 0.1 | 8.29E-07 | 1.52298 E-05 | 9.69E-10 | 5.80026E-07 |
| 0.3 | 1.63E-07 | 7.42332E-05 | 8.67E-09 | 3.66544E-06 |
| 0.4 | 4.88E-07 | 1.08810E-04 | 8.850E-09 | 5.67330E-06 |
| 0.5 | 4.62E-07 | 1.40256E-04 | 2.52E-09 | 7.58382E-06 |
| 0.7 | 8.12E-07 | 1.68693E-04 | 3.57E-09 | 9.54427E-06 |
| 0.9 | 6.60E-07 | 9.78854E-05 | 7.56E-09 | 5.68883E-06 |
| 1.0 | 0 | 1.50887E-15 | 0 | 1.57219E-15 |

*It is observed that for n=128 the maximum absolute error for the example 3 is 5.64E-11 and the maximum absolute error for n=128 for Example 3 obtained by the method [20] is 8.9762E-11, which shows that present method is better than [20].*

**Table 4. Comparison of numerical results**

| x | Absolute errors (n=30) | Absolute errors (n=50) | Khan and Aziz [21] |
|---|---|---|---|
| 0.0 | 0.000000 | 0.000000 | 0.000000 |
| 0.1 | 2.22E-06 | 3.22E-07 | 0.0000056 |
| 0.2 | 4.67E-06 | 4.75E-07 | 0.0000095 |
| 0.3 | 1.89E-06 | 1.94E-07 | 0.0000032 |
| 0.4 | 3.56E-06 | 6.28E-07 | 0.0000175 |
| 0.5 | 7.43E-07 | 8.11E-07 | 0.0000292 |
| 0.6 | 1.22E-06 | 4.38E-07 | 0.0000288 |
| 0.7 | 2.78E-06 | 3.86E-07 | 0.0000132 |
| 0.8 | 3.74E-06 | 7.77E-07 | 0.0000051 |
| 0.9 | 4.11E-06 | 2.35E-07 | 0 |
| 1.0 | 0 | 0 | 0 |

**Example 5** Consider the following third order nonlinear boundary value problem [22]

$$u^{(3)}(x) - ku^{(1)}(x) + r = 0, \qquad 0 \le x \le 1 \\ u\left(\frac{1}{2}\right) = 0, \ u^{(1)}(0) = 0, \ u^{(1)}(1) = 0.$$ (4.5)

The exact solution of the problem (4.5) is $u(x) = \dfrac{r\left(k(2x-1) - 2\sinh(kx) + 2\cosh(kx)\tanh\left(\dfrac{k}{2}\right)\right)}{2k^3}$.

The comparison of the errors in absolute values between the method developed in this paper and the method developed by Li and Wu [22] is shown in Table 5 and 6 for *r=1, k=5* and *10*. This comparison shows that the present method is effective.

**Table 5 Comparison of numerical results when *k=5***

| x | Absolute errors (n=10) [22] | Present method errors (n=10) | Absolute errors (n=50) [22] | Present method errors (n=50) |
|---|---|---|---|---|
| 0.0 | 2.81 E-05 | 7.66E-06 | 1.04E-06 | 8.37E-07 |
| 0.1 | 2.26E-05 | 8.42E-06 | 8.44E-07 | 3.39E-07 |
| 0.2 | 1.41E-05 | 9.25E-06 | 5.27E-07 | 9.16E-08 |
| 0.3 | 7.32E-06 | 1.83E-06 | 2.74E-07 | 7.22E-08 |
| 0.4 | 2.95E-06 | 9.78E-06 | 1.10E-07 | 7.86E-08 |
| 0.6 | 2.95E-06 | 9.92E-06 | 1.10E-07 | 6.35E-08 |
| 0.7 | 7.32E-06 | 3.17E-06 | 2.74 E-07 | 6.26E-08 |
| 0.8 | 1.41E-05 | 7.20E-06 | 5.24 E-07 | 9.54E-08 |
| 0.9 | 2.26E-05 | 8.35E-06 | 8.44E-07 | 3.37E-07 |
| 1.0 | 2.81E-05 | 9.54E-06 | 1.04E-06 | 8.48E-07 |

**Table 6. Comparison of numerical results when *k=10***

| x | Absolute errors (n=10) [22] | Present method errors (n=10) | Absolute errors (n=50) [22] | Present method errors (n=50) |
|---|---|---|---|---|
| 0.0 | 1.14 E-05 | 8.66E-06 | 4.27 E-07 | 9.65E-08 |
| 0.1 | 5.73 E-06 | 1.42E-06 | 1.60E-07 | 7.47E-08 |
| 0.2 | 8.65 E-07 | 1.25E-07 | 4.77E-08 | 1.93E-08 |
| 0.3 | 2.22 E-06 | 7.83E-07 | 8.80E-08 | 2.16E-08 |
| 0.4 | 1.39E-06 | 9.78E-07 | 5.40E-08 | 1.24E-08 |
| 0.6 | 1.39E-06 | 8.92E-07 | 5.40E-08 | 8.69E-08 |
| 0.7 | 2.22E-06 | 7.17E-07 | 8.80E-08 | 1.03E-08 |
| 0.8 | 8.65E-07 | 2.20E-07 | 4.77 E-08 | 1.10E-08 |
| 0.9 | 5.72E-06 | 9.35E-07 | 1.60E-07 | 6.96E-08 |
| 1.0 | 1.14E-05 | 7.54E-06 | 4.27E-07 | 8.32E-08 |

**Example 6** Consider the following third order nonlinear boundary value problem [23]

$$u^{(3)}(x) - \frac{e^{\frac{x}{2}}}{2}u^{(2)}(x) - \sin\sqrt{x}e^x u(x) = f(x), \qquad 0 \le x \le 1 \\ u(0) = 0, \ u\left(\frac{1}{2}\right) = 0, \ u(1) = 0.$$ (4.6)

where $f(x) = 6 + e^{\frac{x}{2}}(3x - \frac{3}{2}) - e^x x(1 - 3x + 2x^2)\sin\sqrt{x}$. The exact solution of the problem (4.6) is $u(x) = x(x - \frac{1}{2})(x - 1)$.

It is observed that the maximum absolute error obtained by the method developed for *n=11* is 1.11E-07 and for *n= 51* is 9.14E-09. It can easily be seen from the Fig. 1 of the paper [23] that our results are better than that of [23].

# 5      Conclusion

In this paper, the solution of linear and nonlinear third order (two and three point) boundary value problem is determined. For the solution of third order (two and three point) boundary value problem reproducing kernel method is proposed and obtained a good accuracy in absolute errors. As the reproducing kernel method cannot solve the third order three-point boundary value problems directly, so the third order boundary value problem is converted to second order two point boundary value problem after absorbing the nonlocal condition at $\alpha$. The method developed is compared with those developed by Li et al. [19], El-Salam et al. [20], Khan and Aziz [21], Li and Wu [22] and Wu and Li [23]. As observed in Example 3, 4, 5 and 6 that the results obtained in this paper are better than [19-23].

# Competing Interests

Authors have declared that no competing interests exist.

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