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On a generalization of KU-algebras pseudo-KU algebras

Daniel A. Romano

International Mathematical Virtual Institute 6, Kordunaška Street, 78000 Banja Luka, Bosnia and Herzegovina.; bato49@hotmail.com

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Abstract: As a generalization of KU-algebras, the notion of pseudo-KU algebras is introduced in 2020 by the author (D. A. Romano. *Pseudo-UP algebras, An introduction*. Bull. Int. Math. Virtual Inst., 10(2)(2020), 349-355). Some characterizations of pseudo-KU algebras are established in that article. In addition, it is shown that each pseudo-KU algebra is a pseudo-UP algebra. In this paper it is a concept developed of pseudo-KU algebras in more detail and it has identified some of the main features of this type of universal algebras such as the notions of pseudo-subalgebras, pseudo-ideals, pseudo-filters and pseudo homomorphisms. Also, it has been shown that every pseudo-KU algebra is a pseudo-BE algebra. In addition, a congruence was constructed on a pseudo-KU algebra generated by a pseudo-ideal and shown that the corresponding factor-structure is and pseudo-KU algebra as well.

Keywords: KU-algebra, Pseudo-KU algebra, pseudo-UP algebra, pseudo-BE algebra, pseudo-ideals, pseudo-homomorphism.

MSC: 62D05.

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1. Introduction

The concept of pseudo-BCK algebras was introduce in [1] by Georgescu and Iorgulescu as an extension of BCK-algebras. The notion of pseudo-BCI algebras was introduced and analyzed in [2] by Dudek and Jun as a generalization of BCI-algebras. The concept of pseudo-BE algebras was introduced in 2013 and their properties were explored by Borzooei *et al.*, in [3]. These algebraic structures has been in the focus of many authors (for example, see [4–10]). Pseudo BL-algebras are a non-commutative generalization of BL-algebras introduced in [11]. Pseudo BL-algebras are intensively studied by many authors (for example, [12–14]).

Prabpayak and Leerawat 2009 in [15,16] introduced a new algebraic structure which is called KU-algebras. They studied ideals and congruences in KU-algebras. They also introduced the concept of homomorphism of KU-algebras and investigated some related properties. Moreover, they derived some straightforward consequences of the relations between quotient KU-algebras and isomorphism. Many authors took part in the study of this algebraic structure (for example: [17,18]).

A detailed listing of the researchers and their contributions to these activities it can be found in [19]. Here, we will highlight the contribution of [20]. In [21], Kim and Kim introduced the concept of BE-algebras as a generalization of dual BCK-algebras. This class of algebra was also studied by Rezaei and Saeid 2012 in article [22]. In the article [20], the authors (Rezaei, Saeod and Borzooei) proved that a KU-algebra is equivalent to a commutative self-distributive BE-algebra. (A BE-algebra A is a self-distributive if $x \cdot (y \cdot z) = (z \cdot y) \cdot (x \cdot z)$ for all $x, y, z \in A$.) Additionally, they proved that every KU-algebra is a BE-algebra ([20], Theorem 3.4), every Hilbert algebra is a KU-algebra ([20], Theorem 3.5) and a self-distributive KU-algebra is equivalent to a Hilbert algebra ([20]). Iampan constructed PU-algebra as a generalization of KU-algebra in [19] in 2017 and showed that each KU-algebra is a PU-algebra.

In article [23], the author designed the concepts of pseudo-UP ([23], Definition 3.1) and pseudo-KU-algebras ([23], Definition 4.1) and showed that each pseudo-KU algebra is a pseudo-UP algebra ([23], Theorem 4.1). However, the term 'pseudo KU-algebra' and mark 'PKU' has already been used in [24] for different purposes. It should be noted here that this term 2019 has been renamed to 'JU-algebra' ([25]). Although introducing the term 'pseudo-KU algebra' as a name for a structure constructed in the manner described here and using the abbreviation 'pKU' for this algebra can lead to confusion, we did it for needs of article [23] and of this paper.

In this paper we develop the concept in more detail of pseudo-KU algebras and we identify some of the main features of this type of universal algebras. The paper was designed as follows: After the Section 2, which outlines the necessary previous terms, Section 3 introduces the concept of pseudo-KU algebra and analyzes some of its important properties. In Section 4, the concept of pseudo-KU algebras is linked to the concepts of pseudo-UP and pseudo-BE algebras. Section 5 deals with some substructures of this class of algebras such as pseudo-subalgebras, pseudo-ideals and pseudo-filters. Finally, in Section 6, the concepts of pseudo-homomorphisms and congruences on pseudo-KU algebras are analyzed.

2. Preliminaries

In this section we will describe some elements of KU-algebras from the literature [15,16] necessary for our intentions in this text.

Definition 1. ([15]) An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a KU-*algebra* where A is a nonempty set, $'\cdot'$ is a binary operation on A, and 0 is a fixed element of A (i.e. a nullary operation) if it satisfies the following axioms:

(KU-1) $(\forall x, y, z \in A)((x \cdot y) \cdot ((y \cdot z) \cdot (x \cdot z)) = 0)$,
(KU-2) $(\forall x \in A)(0 \cdot x = x)$,
(KU-3) $(\forall x \in A)(x \cdot 0 = 0)$, and
(KU-4) $(\forall x, y \in A)((x \cdot y = 0 \ \wedge \ y \cdot x = 0) \implies x = y)$.

On a KU-algebra $A = (A, \cdot, 0)$, we define the KU-ordering $\leqslant$ on A as follows ([15], pp. 56):

$$(\forall x, y \in A)(x \leqslant y \iff y \cdot x = 0).$$

Lemma 1. *In a KU-algebra A, the following properties hold:*

(1) $(\forall x \in A)(x \leqslant x)$,
(2) $(\forall x, y \in A)((x \leqslant y \wedge y \leqslant x), \implies x = y)$,
(3) $(\forall x, y, z \in A)((x \leqslant y \wedge y \leqslant z) \implies x \leqslant z)$,
(4) $(\forall x, y, z \in A)(x \leqslant y \implies z \cdot x \leqslant z \cdot y)$,
(5) $(\forall x, y, z \in A)(x \leqslant y \implies y \cdot z \leqslant x \cdot z)$,
(6) $(\forall x, y \in A)(x \cdot y \leqslant y)$ *and*
(7) $(\forall x \in A)(0 \leqslant x)$.

Definition 2. ([15]) Let S be a non-empty subset of a KU-algebra A.

(a) The subset S is said to be a KU-*subalgebra* of A if $(S, \cdot, 0)$ is a KU-algebra.
(b) The subset S is said to be an *ideal* of A if it satisfies the following conditions:
(J1) $0 \in S$, and
(J2) $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in S \wedge y \in S) \implies x \cdot z \in S)$.

As shown in [18], this kind of algebra satisfies one specific equality.

Lemma 2 ([18])**.** *In a KU-algebra A, the following holds:*

(KU-5) $(\forall x, y, z \in A)(z \cdot (y \cdot x) = y \cdot (z \cdot x))$.

In the light of the previous equality, condition (J2) is transformed into condition:
(J3) $(\forall x, y \in A)((x \cdot y \in S \wedge x \in S) \implies y \in S)$.
Indeed, if we put $x = 0, y = x$ and $z = y$ in (J2), we immediately obtain (J3) by (KU-2). Conversely, let (J3) be a valid formula and let $x, y, z \in A$ be arbitrary elements such that $x \cdot (y \cdot z) \in J$ and $y \in J$. Then $y \cdot (x \cdot z) \in J$ by (KU-5). Thus $x \cdot z \in J$ by (J3).
From (J3) it immediately follows:

Lemma 3. *Let S be an ideal in a KU-algebra A. Then*

(J4) $(\forall x, y \in A)((x \leqslant y \wedge y \in S) \implies x \in S)$.

We can introduce the concept of filters in KU-algebra if formula (J3) serves as a motivation.

Definition 3. The subset F is said to be a *filter* of A if it satisfies the following conditions:

(F1) $0 \in F$, and

(F3) $(\forall x, y \in A)((x \cdot y \in F \wedge y \in F) \implies x \in F)$.

A filter in KU-algebra, designed in this way, has the following property:

Lemma 4. *Let F be a filter in a KU-algebra A. Then*

(F4) $(\forall x, y \in A)((x \leqslant y \wedge x \in F) \implies y \in F)$.

3. Concept of pseudo-KU algebra

Definition 4. ([23]) An algebra $\mathfrak{A} = ((A, \leqslant), \cdot, *, 0)$ of type $(2, 2, 0)$ is called a *pseudo-KU algebra* if it satisfies the following axioms:

(pKU-1): $(\forall x, y, z \in A)((y \cdot x) \leqslant ((x \cdot z) * (y \cdot z)) \wedge (y * x) \leqslant ((x * z) \cdot (y * z)))$,

(pKU-2): $(\forall x \in A)((0 \cdot x = x) \wedge (0 * x = x))$,

(pKU-3): $(\forall x \in A)(x \leqslant 0)$,

(pKU-4): $(\forall x, y \in A)((x \leqslant y \wedge y \leqslant x) \implies x = y)$, and

(pKU-5): $(\forall x, y \in A)((x \leqslant y \iff x \cdot y = 0) \wedge (x \leqslant y \iff x * y = 0))$.

Remark 1. We emphasize that in pseudo-KU algebra the relation of the order is determined inversely with respect to the definition of the order in the KU-algebra.

Lemma 5. *If $\mathfrak{A}$ is a pseudo-KU algebra, then*

(pKU-6) $(\forall x \in A)((x \cdot x = 0) \wedge (x * x = 0))$.

Proof. If we put $x = 0$, $y = 0$, and $z = x$ in the formula (pKU-1), we get

$$(0 \cdot 0) * ((0 \cdot x) \star (0 \cdot x)) = 0 \ \wedge \ (0 * 0) \cdot ((0 * x) \cdot (0 * x)) = 0.$$

From where we get

$$x \cdot x = 0 \ \wedge \ x * x = 0$$

with respect to (pKU-2). □

Proposition 1. *If $\mathfrak{A}$ is a pseudo-KU algebra, then*

(11) $(\forall x, y, z \in A)(x \leqslant y \implies ((y \cdot z \leqslant x \cdot z) \wedge (y * z \leqslant x * z))$ *and*

(12) $(\forall x, y, z \in A)(x \leqslant y \implies ((z \cdot x \leqslant z \cdot y) \wedge (z * x \leqslant z * y))$.

Proof. Let $x, y, z \in A$ such that $x \leqslant y$. Then $x \cdot y = 0 = x * y$. If we put $x = y$ and $y = x$ in (pKU-1), we get

$$0 = (x \cdot y) * ((y \cdot z) * (x \cdot z)) = 0 * ((y \cdot z) * (x \cdot z)) = (y \cdot z) * (x \cdot z).$$

So, we have $y \cdot z \leqslant x \cdot z$. Similarly, we have

$$0 = (x * y) \cdot ((y * z) \cdot (x * z)) = 0 \cdot ((y * z) \cdot (x * z)) = (y * z) \cdot (x * z)$$

and $y * z \leqslant z * x$.

On the other hand, if we put $z = y$ and $y = z$ in (pKU-1), we have

$$0 = (z \cdot x) * ((x \cdot y) * (z \cdot y))) = (z \cdot x) * ((0 * (z \cdot y)) = (z \cdot x) * (z \cdot y).$$

This means $z \cdot x \leqslant z \cdot y$. It can be similarly proved that it is $z * x \leqslant z * y$. □

In 2011, Mostafa, Naby and Yousef proved Lemma 2.2 in [18]. In the following Proposition, we show that analogous equality is also valid in pseudo-KU algebras.

Proposition 2. *In pseudo-KU algebra $\mathfrak{A}$, then*

(pKU) $(\forall x, y, z \in A)(x * (y \cdot z) = y * (x \cdot z) \ \wedge \ x \cdot (y * z) = y \cdot (x * z))$

is valid formula.

Proof. If we put $y = 0$ in (pKU-1), we have

$$0 \cdot x \leqslant (x \cdot z) * (0 \cdot z).$$

Then, we have $x \leqslant (x \cdot z) * z$. From here it follows

$$((x \cdot z) * z) \cdot (y * z) \leqslant x \cdot (y * z)$$

by (11). On the other hand, if we put $x = z \cdot z$ in (pKU-1), we get

$$y * (x \cdot z) \leqslant ((x \cdot z) * z) \cdot (y * z) \leqslant x \cdot (y * z).$$

Since the variables $x, y, z \in A$ are free variables, if we put $x = y$ and $y = x$, we get an inverse inequality. From here it follows (pKU) by (pKU-4).

The other equality can be proved in an analogous way. □

4. Correlation of pseudo-KU algebras with other types of pseudo algebras

The notion of pseudo-UP algebra as a generalization of the concept of UP-algebras was introduced and analyzed in [23].

Definition 5. ([23]) A *pseudo-UP algebra* is a structure $\mathfrak{A} = ((A, \leqslant), \cdot, *, 0)$, where $' \leqslant '$ is a binary relation on a set A, $' \cdot '$ and $' * '$ are internal binary operations on A and $'0'$ is an element of A, verifying the following axioms:

(pUP-1) $(\forall x, yz \in A)(y \cdot z \leqslant (x \cdot y) * (x \cdot z) \ \wedge \ y * z \leqslant (x * y) \cdot (x * z))$;
(pUP-4) $(\forall x, y \in A)((x \leqslant y \ \wedge \ y \leqslant x) \implies x = y)$;
(pUP-5) $(\forall x, y \in A)((y \cdot 0) * x = x \ \wedge \ (y * 0) \cdot x = x)$ and
(pUP-6) $(\forall x, y \in A)((x \leqslant y \iff x \cdot y = 0) \ \wedge \ (x \leqslant y \iff x * y = 0))$.

The following theorem is an important result of pseudo-KU algebras for study in the connections between pseudo-UP algebras and pseudo-KU algebras.

Theorem 1. *Any pseudo-KU algebra is a pseudo-UP algebra.*

Proof. It only needs to show (pUP-1). By Proposition 2, we have that any pseudo-KU algebra satisfies (pUP-1). □

Pseudo-BE algebra is defined by the follows:

Definition 6. ([3]) An algebra $A = (A, \cdot, *, 1)$ of type $(2, 2, 0)$ is called a *pseudo BE-algebra* if satisfies in the following axioms:

(pBE-1) $(\forall x \in A)(x \cdot x = 1 \ \wedge \ x * x = 1)$;
(pBE-2) $(\forall x \in A)(x \cdot 1 = 1 \ \wedge \ x * 1 = 1)$;
(pBE-3) $(\forall x \in A)(1 \cdot x = x \ \wedge \ 1 * x = x)$;
(pBE-4) $(\forall x, y, z \in A)(x \cdot (y * z) = y * (x \cdot z))$; and
(pBE-5) $(\forall x, y \in A)(x \cdot y = 1 \iff x * y = 1)$.

If we replace 1 with 0 in (BE-1), (BE-2), (BE-3) and (BE-5) and prove that the formula (pBE-4) is a valid formula in a pseudo-KU algebra A, we have proved that every pseudo-KU algebra A is a pseudo-BE algebra.

Theorem 2. *Any pseudo-KU algebra is a pseudo-BE algebra.*

Proof. It is sufficient to prove that the formula (pBE-4) is a valid formula in any pseudo-KU algebra. If we put $y = 0$ in the left-hand side of the formula (pKU-1), we get $0 \cdot x \leqslant ((x \cdot z) * (0 \cdot z)$. It means $x \leqslant (x \cdot z) * z$. From here follows

$$((x \cdot z) * z) \cdot (y * z) \leqslant x \cdot (y * z),$$

by the left part of formula (11). On the other hand, if we put $x = x \cdot z$ in the right-hand side of the formula (pKU-1), we get

$$y * (x \cdot z) \leqslant ((x \cdot z) * z) \cdot (y * z).$$

Which together with the previous inequality gives

$$y * (x \cdot z) \leqslant x \cdot (y * z).$$

From this inequality by substituting the variables x and y, we obtain the necessary reverse inequality

$$x \cdot (y * z) \leqslant y * (x \cdot z).$$

From these two inequalities follows the validity of the formula (pBE-4) in any pseudo-KU algebra by the axiom (pKU-4). □

Since the formula previously proven is important below, we point it out in particular.

Proposition 3. *In any pseudo-KU algebra* $\mathfrak{A}$,
(pKU-7) $(\forall x, y, z \in A)(x \cdot (y * z) = y * (x \cdot z))$
is a valid formula.

5. Some substructures in pseudo-KU algebras

5.1. Concept of pseudo-subalgebras

Definition 7. A nonempty subset S of a pseudo-KU algebra A is a *pseudo-subalgebra* in $\mathfrak{A}$ if

$$(\forall x, y \in A)((x \in S \wedge y \in S) \implies (x \cdot y \in S \wedge x * y \in S)).$$

holds.

Putting $y = x$ in the previous definition, it immediately follows:

Lemma 6. *If* S *is a pseudo-subalgebra of a pseudo-KU algebra* $\mathfrak{A}$, *then* $0 \in S$.

Proof. Let S be a pseudo-subalgebra of a pseudo-KU algebra $\mathfrak{A}$. It means that S is a nonempty subset of A. Then there exists an element $y \in S$. Thus $0 = y \cdot y = y * y \in S$ by Definition 7. □

It is clear that subsets $\{0\}$ and A are pseudo-subalgebras of a pseudo-KU algebras $\mathfrak{A}$. So, the family $\mathfrak{S}(A)$ of all pseudo-subalgebras of a pseudo-KU algebra $\mathfrak{A}$ is not empty. Without major difficulties, the following theorem can be proved.

Theorem 3. *The family* $\mathfrak{S}(A)$ *of all pseudo-subalgebras of a pseudo-KU algebra* $\mathfrak{A}$ *forms a complete lattice.*

5.2. Concept of pseudo-ideals

Definition 8. The subset J is said to be a *pseudo-ideal* of a pseudo-KU algebra $\mathfrak{A}$ if it satisfies the following conditions:
(pJ1) $0 \in J$,
(pJ3a) $(\forall x, y \in A)((x \cdot y \in J \wedge x \in J) \implies y \in J)$ and
(pJ3b) $(\forall x, y \in A)((x * y \in J \wedge x \in J) \implies y \in J)$.

Proposition 4. *Let* J *be a nonempty subset of a pseudo-KU algebra* $\mathfrak{A}$. *Then the condition* $(pJ3a)$ *is equivalent to the condition:*
(pJ4a) $(\forall x, y, z \in A)((x * (y \cdot z) \in J \wedge y \in J) \implies x * z \in J)$.

Proof. Putting $x = y$ and $y = x * z$ in the condition (pJ3a), it immediately follows

$$(\forall x, y, z \in A)((y \cdot (x * z) \in J \wedge y \in J) \implies x * z \in J).$$

Thus
$$(\forall x, y, z \in A)((y * (x \cdot z) \in J \wedge y \in J) \implies x * z \in J)$$
by (pKU-7).

Conversely, let (pJ4a) it be. Let us choose $x = 0$, $y = x$ and $z = y$ in (pJ4a). We get $(0 * (x \cdot y) \in J \wedge x \in J) \implies 0 * y \in J$. Thus (pJ3a) by (pKU-2). □

Corollary 4. *Let J be a pseudo-ideal in a pseudo-KU-algebra $\mathfrak{A}$. Then*
(13) $(\forall x, y \in A)(y \in J \implies x * y \in J)$.

Proof. Putting $z = y$ in (pJ4a), with respect to (pKU-6), (pKU-3) and (pJ1), we obtain (13). □

Proposition 5. *Let J be a nonempty subset of a pseudo-KU algebra $\mathfrak{A}$. Then the condition $(pJ3b)$ is equivalent to the condition*
(pJ4b) $(\forall x, y, z \in A)((x \cdot (y * z) \in J \wedge y \in J) \implies x \cdot z)$.

Proof. If we put $x = y$ and $y = x \cdot z$ in (pJ3b), we get
$$(y * (x \cdot z) \in J \wedge y \in J) \implies x \cdot z \in J.$$
Hence
$$(x \cdot (x * z) \in J \wedge y \in J) \implies x \cdot z \in J.$$
by (pKU-7).

Conversely, if we put $x = 0$, $y = x$, and $z = y$ in (pJ4b), we get
$$(0 \cdot (x * y) \in J \wedge x \in J) \implies 0 \cdot y \in J.$$
Thus (pJ3b) with respect to (pKU-2). □

Corollary 5. *Let J be a pseudo-ideal in a pseudo-KU-algebra $\mathfrak{A}$. Then*
(14) $(\forall x, y \in A)(y \in J \implies x \cdot y \in J)$.

Proof. Putting $z = y$ in (pJ4b), with respect to (pKU-6), (pKU-3) and (pJ1), we obtain (14). □

The following important statement describes the connection between conditions (pJ3a) and (pJ3b).

Proposition 6. *Let J be a pseudo-ideal of a pseudo-KU algebra $\mathfrak{A}$. Then*
$$(pJ3a) \iff (pJ3b).$$

Proof. $(pJ3a) \iff (pJ3b)$**.** Suppose (pJ3a) holds and let $x * y \in J$ and $x \in J$. How obvious it is that the following
$$x * ((x \cdot y) * y) = 0 \iff x \cdot ((x \cdot y) \star x) = 0 \iff (x \cdot y) * (x \cdot y) = 0$$
is valid, we have
$$(x \in J \wedge x \cdot ((x * u) \cdot y) = 0 \in J) \implies (x * y) \cdot y \in J.$$
Now
$$(x * y \in J \wedge (x * y) \cdot y \in J) \implies y \in J.$$
We have proved that (pJ3b) is a valid implication.

$(pJ3b) \implies (pJ3a)$**.** Let (pJ3b) be a valid formula and let $x, y \in A$ be such that $x \in J$ and $x \cdot y \in J$. As above, from
$$x * ((x \cdot y) * y) = 0 \iff x \cdot ((x \cdot y) * y) = 0 \iff (x \cdot y) * (x \cdot y) = 0$$
it follows
$$(x \in J \wedge x * ((x \cdot y) * y) = 0 \in J) \implies (x \cdot y) * y \in J.$$
Now, $x \cdot y \in J$ and $(x \cdot y) * y$ it follows $y \in J$. This proves the validity of the formula (pJ3a). □

Proposition 7. *Any pseudo-ideal in a pseudo-KU-algebra $\mathfrak{A}$ is a pseudo-subalgebra in $\mathfrak{A}$.*

Proof. The proof of this proposition follows from (13) and (14). □

Theorem 6. *The family $\mathfrak{J}(A)$ of all pseudo-ideals in a pseudo-KU algebra $\mathfrak{A}$ forms a complete lattice and $\mathfrak{J}(A) \subseteq \mathfrak{S}(A)$ holds.*

Proof. Let $\{J_i\}_{i\in I}$ be a family of pseudo-ideals in a pseudo-KU algebra $\mathfrak{A}$. Clearly $0 \in \bigcap_{i\in I} J_i$ is valid. Let $x, y \in A$ be elements such that $x \cdot y \in \bigcap_{i\in I} J_i$, $x * y \in \bigcap_{i\in I} J_i$ and $x \in \bigcap_{i\in I} J_i$. Then $x \cdot y \in J_i$, $x * y \in J_i$ and $x \in F_i$ for any $i \in I$. Thus $y \in J_i$ because J_i is a pseudo-ideal in $\mathfrak{A}$ and $x \in \bigcap_{i\in I} J_i$. So, $\bigcap_{i\in I} J_i$ is a pseudo-ideal in $\mathfrak{A}$.

If $\mathfrak{X}$ is the family of all pseudo-ideals of $\mathfrak{A}$ that contain the union $\bigcup_{i\in I} J_i$, then $\cap\mathfrak{X}$ is also a pseudo-ideal in $\mathfrak{A}$ that contains $\bigcup_{i\in I} J_i$ by previous evidence.

If we put $\sqcap_{i\in I} J_i = \bigcap_{i\in I} J_i$ and $\sqcup_{i\in I} J_i = \cap\mathfrak{X}$, then $(\mathfrak{J}(A), \sqcap, \sqcup)$ is a complete lattice. □

To round out this subsection we need the following lemma.

Lemma 7. *Let J be a pseudo-ideal in a pseudo-KU algebra $\mathfrak{A}$. Then*
(15) $(\forall x, y \in A)((x \leqslant y \wedge x \in J) \implies y \in J)$.

Proof. The proof of this proposition follows from (pJ3a) (or (pJ3b)) with respect to (pKU-6) and (pJ1). □

Theorem 7. *Let J be a subset of a pseudo-KU algebra $\mathfrak{A}$ such that $0 \in J$. Then, J is a pseudo-ideal in $\mathfrak{A}$ if and only if the following holds*
(pJ5) $(\forall x, y, z \in A)((x \in A \wedge y \in A \wedge x \leqslant y \cdot z) \implies z \in J)$.

Proof. Let J be a pseudo-ideal in $\mathfrak{A}$ and let $x, y, z \in A$ such that $x \in J$, $y \in J$ and $x \leqslant y \cdot z$. Then $x \cdot (y \cdot z) = 0 \in J$. Thus $y \cdot z \in J$ by (pJ3a) and again, from here and $y \in J$ it follows $z \in J$. So, we have shown that (pJ5) is a valid formula.

Opposite, suppose that (pJ5) is a valid in $\mathfrak{A}$. Let us show that J is a pseudo-ideal and $\mathfrak{A}$. Let $x, y \in A$ be such that $x \in J$ and $x \cdot y \in J$. Then $x * y \in J$ by Proposition 6. On the other hand, from $x \cdot ((x * y) \cdot y) = 0$, i.e. from $x \leqslant (x * y) \cdot y$ it follows $y \in J$ by hypothesis. So, the set J is a pseudo-ideal in $\mathfrak{A}$. □

For a relation on the set A we say that it is a quasi-order relation on A if it is reflexive and transitive. It is easy to prove that if σ is a quasi-order relation on A, then the relation $\sigma \cap \sigma^{-1}$ is an equivalence on A.

Theorem 8. *Let J be a pseudo-ideal in a pseudo-KU algebra $\mathfrak{A}$. Then the relation $' \preccurlyeq '$, defined by*

$$(\forall x, y \in A)(x \preccurlyeq y \iff x \cdot y \in J),$$

is a quasi-order in the set A left compatible and right reverse compatible with the internal operations in $\mathfrak{A}$.

Proof. Since $x \cdot x = 0 \in J$ is valid in $\mathfrak{A}$ for any $x \in A$, it is clear that $' \preccurlyeq '$ is a reflexive relation in the set A.

Let $x, y, z \in A$ be arbitrary elements such that $x \preccurlyeq y$ and $y \preccurlyeq z$. This means $x \cdot y \in J$ and $y \cdot z \in J$. From inequality (pKU-1) in the form $x \cdot y \leqslant (y \cdot z) * (x \cdot z)$ and $x \cdot y \in J$ it follows $(y \cdot z) * (x \cdot z) \in J$ according to (15). From here and from $y \cdot z \in J$ it follows $x \cdot z \in J$ according to (pJ3a). Hence, the relation $' \preccurlyeq '$ is transitive. So, this relation is a quasi-order in A.

Let $x, y, z \in A$ be such $x \preccurlyeq y$. Then $x \cdot y \in J$ and $x * y \in J$.

(i) If we put $x = y$ and $y = x$ in the left part of the formula (pKU-1), we get $x \cdot y \leqslant (y \cdot z) * (x \cdot z)$. Now, from here and $x \cdot y \in J$ it follows $(y \cdot z) * (x \cdot z) \in J$ by (15). Thus $(y \cdot z) \cdot (x \cdot z) \in J$ by Proposition 6. Finally, we have $y \cdot z \preccurlyeq x \cdot z$. So, the relation $' \preccurlyeq '$ is reverse right compatible with the internal operation $' \cdot '$ in $\mathfrak{A}$.

(ii) If we put $x = y$ and $y = x$ in the right part of the formula (pKU-1), we get $x * y \leqslant (y * z) \cdot (x * z)$. Then $(y * z) \cdot (x * z) \in J$ by (15). Thus $y * z \preccurlyeq x * z$. Therefore, the relation $' \preccurlyeq '$ is reverse right compatible with the internal operation $' * '$ in $\mathfrak{A}$.

(iii) Let us put $y = z$ and $z = y$ in the left part of the formula (pKU-1). We get $(z \cdot x) * ((x \cdot y) * (z \cdot y)) = 0 \in J$. From here and from $x \cdot y \in J$ it follows $(z \cdot x) * (z \cdot y) \in J$ by (pJ4a). Thus $z \cdot x \preccurlyeq z \cdot y$. So, the relation $' \preccurlyeq '$ is left compatible with the operation $' \cdot '$.

(iv) Let us put $y = z$ and $z = y$ in the right part of the formula (pKU-1). We get $(z * x) \cdot ((x * y) \cdot (z * y)) = 0 \in J$. From here and from $x * y \in J$ it follows $(z * x) \cdot (z * y) \in J$ by (pJ4b). Thus $z * x \preccurlyeq z * y$. So, the relation $' \preccurlyeq '$ is left compatible with the operation $' * '$. □

5.3. Concept of pseudo-filters

Definition 9. A non-empty subset F of a pseudo-KU algebra $\mathfrak{A}$ is called a *pseudo-filter* of A if it satisfies in the following axioms:
(pF1) $0 \in F$;
(pF3) $(\forall x, y \in A)((x \cdot y \in F \wedge x * y \in F \wedge y \in F) \Longrightarrow x \in F)$.

$\{0\}$ and A are pseudo-filters of $\mathfrak{A}$. So, the family $\mathfrak{F}(A)$ of all pseudo-filters in a pseudo-KU algebra $\mathfrak{A}$ is not empty.

It is obviously the following is valid

Lemma 8. *Let F be a pseudo-filter in a pseudo-KU algebra $\mathfrak{A}$. Then*
(16) $(\forall x, y \in A)((x \leqslant y \wedge y \in F) \Longrightarrow x \in F)$.

Theorem 9. *The family $\mathfrak{F}(A)$ of all pseudo-ideals in a pseudo-KU algebra $\mathfrak{A}$ forms a complete lattice.*

Proof. Let $\{F_i\}_{i \in I}$ be a family of pseudo-filters in a pseudo-KU algebra $\mathfrak{A}$. Clearly $0 \in \bigcap_{i \in I} F_i$ is valid. Let $x, y \in A$ be elements such that $x \cdot y \in \bigcap_{i \in I} F_i$, $x * y \in \bigcap_{i \in I} F_i$ and $y \in \bigcap_{i \in I} F_i$. Then $x \cdot y \in F_i$, $x * y \in F_i$ and $y \in F_i$ for any $i \in I$. Thus $x \in F_i$ because F_i is a pseudo-filter in $\mathfrak{A}$ and $x \in \bigcap_{i \in I} F_i$. So, $\bigcap_{i \in I} F_i$ is a pseudo-filter in $\mathfrak{A}$.

If $\mathfrak{X}$ is the family of all pseudo-filters of $\mathfrak{A}$ that contain the union $\bigcup_{i \in I} F_i$, then $\cap \mathfrak{X}$ is also a pseudo-filter in $\mathfrak{A}$ that contains $\bigcup_{i \in I} F_i$ by previous evidence.

If we put $\sqcap_{i \in I} F_i = \bigcap_{i \in I} F_i$ and $\sqcup_{i \in I} F_i = \cap \mathfrak{X}$, then $(\mathfrak{F}(A), \sqcap, \sqcup)$ is a complete lattice. □

6. Concept of pseudo-homomorphisms

Definition 10. $((A, \leqslant_A), \cdot_A, *_A, 0_A)$ and $((B, \leqslant_B), \cdot_B, *_B, 0_B)$ be pseudo-KU algebras. A mapping $f : A \longrightarrow B$ of pseudo-KU algebras is called a *pseudo-homomorphism* if

$$(\forall x, y \in A)(f(x \cdot_A y) =_B f(x) \cdot_B f(y) \wedge f(x *_A y) =_B f(x) *_B f(y)).$$

Remark 2. Note that if $f : A \longrightarrow B$ is a pseudo homomorphism, then $f(0_A) = 0_B$. Indeed, if we chose $y = x$, from the previous formula we immediately get $f(0_A) =_B 0_B$ with respect (pKU-6).

From here it immediately follows:

Lemma 9. *Any pseudo-homomorphism between pseudo-KU algebras is isotone mapping.*

Proof. Let $f : A \longrightarrow B$ be a pseudo-homomorphism between pseudo-KU algebras and let $x, y \in A$ be such $x \leqslant_A y$. Then $x \cdot_A y =_A 0_A$. Thus $0_B =_B f(x \cdot_A y) =_B f(x) \cdot_B f(y)$. This means $f(x) \leqslant_B f(y)$. □

Lemma 10. *Let $f : A \longrightarrow B$ be a pseudo-homomorphism between pseudo-KU algebras. Then the set $Ker(f) =_A \{x \in A : f(x) =_B 0_B\}$ is a pseudo-ideal in $\mathfrak{A}$.*

Proof. It is obvious $0_A \in Ker(f)$.

Let $x, y \in A$ be such $x \cdot_A y \in Ket(f)$ and $x \in Ker(f)$. Then $f(x) =_B 0_B$ and $0 =_B f(x \cdot_A y) =_B f(x) \cdot_B f(y) =_B 0_B \cdot_B f(y) =_B f(y)$. Thus $y \in Ker(f)$.

The implication of $x *_A y \in Ker(f) \wedge x \in Ker(f) \Longrightarrow y \in Ker(f)$ can be proved by analogy with the previous proof. □

The following statement is easy to prove:

Lemma 11. *If $f : A \longrightarrow B$ is a pseudo-homomorphism between pseudo-KU algebras, then $f(A)$ is a pseudo-subalgebra in B.*

Proposition 8. *Let $f : A \longrightarrow B$ be a pseudo homomorphism between pseudo-KU algebras $\mathfrak{A}$ and $\mathfrak{B}$.*

(i) *If K is a pseudo-ideal in $\mathfrak{B}$, then $f^{-}(K)$ is a pseudo-ideal in $\mathfrak{A}$.*

(ii) *If G is a pseudo-filter in $\mathfrak{B}$, then $f^{-1}(G)$ is a pseudo-filter in $\mathfrak{A}$.*

Proof. (i) Assume that K is a pseudo-fulter of $\mathfrak{B}$. Obviously $0_A \in f^{-1}(K)$. Let $x, y \in A$ be such $x \cdot y \in f^{-1}(K)$ and $x \in f^{-1}(K)$. Then $f(x) \cdot_B f(y) =_B f(x \cdot_A y) \in K$ and $f(x) \in K$. It follows that $f(y) \in K$ by (pJ3a) since K is a pseudo-ideal in $\mathfrak{B}$. Therefore, $y \in f^{-1}(K)$. Thus, the set $f^{-1}(K)$ satisfies the implication (pJ3a). That the set $f^{-1}(K)$ satisfies the implication (pJ3b) can be proved in an analogous way. Therefore, the set $f^{-1}(K)$ is a pseudo-ideal in $\mathfrak{A}$.

(ii) It is obvious $0_A \in f^{-1}(G)$ again. Let $x, y \in A$ be elements such that $x \cdot_A y \in f^{-1}(G)$, $x *_A y \in f^{-1}(G)$ and $y \in f^{-1}(G)$. Then $f(x) \cdot_B f(y) =_B f(x \cdot_A y) \in G$, $f(x) *_B f(y) =_B f(x *_A y) \in G$ and $f(y) \in G$. Thus $f(x) \in G$ because G is a pseudo-filter in $\mathfrak{B}$. This means $x \in f^{-1}(G)$. So, the set$f^{-1}(G)$ is a pseudo-filter in $\mathfrak{A}$. □

In the following definition, we will introduce the concept of congruence on pseudo-KU algebras. Since we have two unitary operations on this algebra, it is possible to determine three different types of congruences.

Definition 11. Let $\mathfrak{A} = ((A, \leqslant), \cdot, *, 0)$ be a pseudo-KU algebra.

For the equivalence relation q on the set A we say that it is a *congruence of type* $' \cdot '$ on $\mathfrak{A}$ if it compatible with the operations $' \cdot '$ in $\mathfrak{A}$ in the following sense

(17) $(\forall x, y, z \in A)((x, y) \in q \implies ((x \cdot z, y \cdot z) \in q \wedge (z \cdot x, z \cdot y) \in q)))$.

For the equivalence relation q on the set A we say that it is a *congruence of type* $' * '$ on $\mathfrak{A}$ if it compatible with the operations $' * '$ in $\mathfrak{A}$ in the following sense

(18) $(\forall x, y, z \in A)((x, y) \in q \implies ((x * z, y * z) \in q \wedge (z * x, z * y) \in q)))$.

For the equivalence relation q on the set A we say that it is a *congruence of common type* on $\mathfrak{A}$ if it is compatible with both operations in $\mathfrak{A}$.

Lemma 12. *Let q be a relation on a pseudo-KU algebra $\mathfrak{A}$. Then:*

(i) *The condition* (17) *is equivalent to the condition*

(17a) $(\forall x, y, u, v \in A)(((x, y) \in q \wedge (u, v) \in q) \implies (x \cdot u, y \cdot v) \in q)$.

(ii) *The condition* (18) *is equivalent to the condition*

(18a) $(\forall x, y, u, v \in A)(((x, y) \in q \wedge (u, v) \in q) \implies (x * u, y * v) \in q)$.

Proof. $(17a) \implies (17)$**.** If we choose $v = z$ in (17a), we get the implication $(x, y) \in q \implies (x \cdot z, y \cdot z) \in q$. On the other hand, if we put $x = y = z$, $u = x$ and $v = y$ in (17a), we get the implication $(x, y) \in q \implies (z \cdot x, z \cdot t)$.

$(17) \implies (17a)$**.** Suppose (17) and let $x, y, u, v \in A$ such that $(x, y) \in q$ and $(u, v) \in q$. Thus $(x \cdot u, x \cdot v) \in q$ and $(x \cdot v, y \cdot v) \in q$ by (16). Hence $(x \cdot u, y \cdot v) \in q$ by transitivity of q.

Equivalence $(18) \iff (18a)$ can be proved analogous to the previous proof. □

Let $f : A \longrightarrow B$ be a pseudo homomorphism between pseudo-KU algebras. By direct check without difficulty, it can be proved that the relation q_f, defined by

$$(\forall x, y \in A)((z, y) \in q_f \iff f(x) =_B f(y)),$$

is a congruence (all three types) on $\mathfrak{A}$.

Theorem 10. *The relation q_f is a congruence of type $' \cdot '$ (type $' * '$, common type) on the pseudo-KU algebra $\mathfrak{A}$.*

Proof. We will only demonstrate the proof that q_f is a congruence of type $' \cdot '$ on $\mathfrak{A}$ because the evidence that q_f is a congruence of type $' * '$ can obtain by analogy with the previous one, and the proof of common type is obtained by combining this two evidences.

Clearly, q_f is an equivalence relation on the set A. It remains to verify that (16) is a valid formula in $\mathfrak{A}$. Let $x, y, u, v \in A$ be such that $(x, y) \in q_f$ and $(u, v) \in q_f$. Then $f(x) =_B f(y)$ and $f(u) =_B f(v)$. Thus

$$f(x \cdot_A u) =_B f(x) \cdot_B f(u) =_B f(y) \cdot_B f(v) =_B f(y \cdot_A u).$$

Hence, $(x \cdot_A u, y \cdot_A v) \in q_f$. We proved that (17a) is a valid formula. So q_f is a congruence of type $' \cdot '$ on $\mathfrak{A}$. $\square$

Theorem 11. *Let J be a pseudo-ideal in a pseudo-KU algebra $\mathfrak{A}$. Then the relation q_J, defined by $q_J = \preccurlyeq \cap \preccurlyeq^{-1}$, is a congruence of common type in $\mathfrak{A}$.*

Proof. The relation q is an equivalence relation on the set A. It is sufficient to prove that q is compatible with operations in $\mathfrak{A}$. Since the relation $\preccurlyeq$ is left compatible and right reverse compatible with the internal operations in $\mathfrak{A}$, by Theorem 8, it is clear that the relation q_J is a congruence on $\mathfrak{A}$. $\square$

For a congruence q on a pseudo-KU algebra $\mathfrak{A}$ we denote $qx = \{y \in A : (x, y) \in q\} = [x]$. Let's define $' \bullet '$ and $' \star '$ in A/q on this way

$$(\forall x, y \in A)([x] \bullet [y] = [x \cdot y]) \text{ and } (\forall x, y \in A)([x] \star [y] = [x * y]).$$

Without much difficulty it can be verified that the functions $' \bullet '$ and $' \star '$, defined in this way, are well-defined internal binary operations in A/q. Also, one can check that the set A/q with the operations $' \bullet '$ and $' \star '$, determined as above, satisfies all the axioms of Definition 4 except the axiom (pKU-4). However, if we take the relation q_J, defined by an pseudo-ideal J of a pseudo-KU algebra $\mathfrak{A}$, then we have

Theorem 12. *Let J be a pseudo-ideal in a pseudo-KU algebra $\mathfrak{A}$. Then the structure $((A/q, \leqq), \bullet, \star, [0])$, where $' \leqq '$ is defined by*

$$(\forall x, y \in A)([x] \leqq [y], \Longleftrightarrow x \preccurlyeq y),$$

is a pseudo-KU algebra, too.

Proof. According to the commentary preceding this theorem, to prove this theorem it suffices to show that the structure $((A/q, \leqq), \bullet, \star, [0])$ satisfies the axiom (pKU-4).

Let $x, y \in A$ be such $[x] \leqq [y]$ and $[y] \leqq [x]$. Then $x \preccurlyeq y$ and $y \preccurlyeq x$ by definition. Thus $(x, y) \in q_J$ and $[x] = [y]$. $\square$

Let $f : A \longrightarrow B$ be pseudo-homomorphism between pseudo-KU algebras $((A, \leqslant_A), \cdot_A, *_A, 0_A)$ and $((B, \leqslant_B), \cdot_B, *_B, 0_B)$. Then the set $f(A)$ is a pseudo-subalgebra of $\mathfrak{B}$ by and the set $J = Ker(f)$ is a pseudo-ideal in $\mathfrak{A}$ by Lemma 10 and the relation q_f is a congruence on $\mathfrak{A}$ by Theorem 10. If $(x, y) \in q_f$ holds soe some $x, y \in A$, we have $f(x) =_B f(y)$. Thus $f(x \cdot_A y) =_B f(x) \cdot_B f(y) =_B f(x) \cdot_B f(x) =_B 0_B$, i.e. $x \cdot_A y \in J$. Analogous to the previous one may be shown that $y \cdot_A x \in J$ holds. Thus, $(x, y) \in q_f \Longrightarrow (x, y) \in q_J$ is valid.

We end this section with the following theorem. Since this theorem can be proven by direct verification, we will omit evidence for it.

Theorem 13. *Let $f : A \longrightarrow B$ be pseudo-homomorphism between pseudo-KU algebras $((A, \leqslant_A), \cdot_A, *_A, 0_A)$ and $((B, \leqslant_B), \cdot_B, *_B, 0_B)$. Then there exists the unique epimorphism $\pi : A \longrightarrow A/q_f$, defined by $\pi(x) = [x]$ for any $x \in A$, and the unique monomorphism $g : A/q_f \longrightarrow B$, defined by $g([x]) =_B f(x)$ for any $x \in A$ such that $f = g \circ \pi$.*

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